1 Introduction

Notes

- **Problem**: binary relationship from inputs to outputs
- **Algorithm**: procedure mapping each input to a single output
  - An algorithm **solves** a problem if it returns a correct output for each and every problem input
- **Correctness**:
  - For small inputs: can use case analysis
  - For arbitrarily large inputs: algorithm either is recursive or loop in some way. Use **induction**.
- **Efficiency**: how fast does an algorithm produce a correct output?
  - Count the number of fixed time operations algorithm takes to return
  - **Asymptotic Notation**: ignore constant factors and low order terms

input	constant	logarithmic	linear	log-linear	quadratic	polynomial	exponential
n	$\Theta(1)$	$\Theta(logn)$	$\Theta(n)$	$\Theta(nlogn)$	$\Theta(n^2)$	$\Theta(n^c)$	$2^{\Theta}(n^c)$
1000	1	≈ 10	1000	$\approx 10,000$	$1,000,000$	1000^c	$2^1000 \approx 10^301$
Time	$1ns$	$10ns$	$1\mu s$	$10\mu s$	$1ms$	$10^(3c-9)s$	$10^281millenia$

- **Model of Computation**: what operations on the machine can be performed in $O(1)$ time.
  - Machine word: block of w bits (w is word size of a w-bit Word-RAM)
  - Memory: Addressable sequence of machine words
  - Processor supports many constant time operations on a $O(1)$ number of words (integers):
    - integer arithmetic: (+, -, *, //, %)
    - logical operators: (&&, ||, !, ==, <, >, <=, =>)
    - bitwise arithmetic: (&, |, <<, >>, ...)
    - Given word a, can read word at address a, write word to address a
- **Data Structure** : a way to store non-constant data, that supports a set of operations
  - A collection of operations is called an **interface**
  - Example:
    - **Sequence**: Extrinsic order to items (first, last, nth)
    - **Set**: Intrinsic order to items (queries based on item keys)
  - Data structures may implement the same interface with different performance
- Example: Static Array - fixed width slots, fixed length, static sequence interface
  - `StaticArray(n)`: allocate static array of size n initialized to 0 in $\Theta(n)$ time
  - `StaticArray.get_at(i)`: return word stored at array index i in $\Theta(1)$ time
  - `StaticArray.set_at(i, x)`: write word x to array index i in $\Theta(1)$ time

More on Asymptotic Notation

- O **Notation**:
  - Non-negative function $g(n)$ is in $O(f(n))$ if and only if there exists a positive real number c and positive integer n_0 such that $g(n) \leq c \cdot f(n)$ for all $n \geq n_0$.
- Ω **Notation**:
  - Non-negative function $g(n)$ is in $\Omega(f(n))$ if and only if there exists a positive real number c and positive integer n_0 such that $c \cdot f(n) \leq g(n)$ for all $n \geq n_0$.
- $\Theta Notation$:

- Non-negative $g(n)$ is in $\Theta(f(n))$ if and only if $g(n) \in O(f(n)) \cap \Omega(f(n))$

2 Data Structures

Notes

Data Structure Interfaces

- A **data structure** is a way to store data, with algorithms that support **operations** on the data
- Collection of supported operations is called an **interface** (also **API** or **ADT**)
- Interface is a **specification**: what operations are supported (the problem!)
- Data structure is a **representation**: how operations are supported (the solution!)

Sequence Interface (L02, L07)

- Maintain a sequence of items (order is **extrinsic**)
- Ex: (x_0, x_1, x_2, $\ldots$, x_{n-1}) (zero indexing)
- (use n to denote the number of items stored in the data structure)
- Supports sequence operations:

Type	Interface	Specification
Container	`build(X)`	given an iterable X, build sequence from items in X
	`len()`	return the number of stored items
Static	`iter_seq()`	return the stored items one-by-one in sequence order
	`get_at(i)`	return the i^{th} item
	`set_at(i, x)`	replace the i^{th} item with x
Dynamic	`insert_at(i, x)`	add x as the i^{th} item
	`delete_at(i, x)`	remove and return the i^{th} item
	`insert_fist(x)`	add x as the first item
	`delete_first(x)`	remove and return the first item
	`insert_last(x)`	add x as the last item
	`delete_last(x)`	remove and return the last item

- Special case interfaces:
  - **stack**: `insert_last(x)` and `delete_last()`
  - **queue**: `insert_last(x)` and `delete_first()`

Set Interface (L03-L08)

- Sequence about **extrinsic** order, set is about **intrinsic** order
- Maintain a set of items having **unique keys** (e.g., item x has key x.key)
- (Set or multi-set? We restrict to unique keys for now.)
- Often we let key of an item be the item itself, but may want to store more info than just key
- Supports set operations:

Type	Interface	Specification
Container	`build(X)`	given an iterable X, build sequence from items in X
	`len()`	return the number of stored items
Static	`find(k)`	return the stored item with key k
Dynamic	`insert(x)`	add x to set (replace item with key x.key if one already exist)
	`delete(x)`	remove and return the stored item with key k
Order	`iter_ord()`	return the stored items one-by-one in key order
	`find_min()`	return the stored item with smallest key
	`find_max()`	return the stored item with largest key
	`find_next(k)`	return the stored item with smallest key larger than k
	`find_prev(k)`	return the stored item with largest key smaller than k

- Special case interfaces:
  - **dictionary**: set without the Order operations

Array Sequence

- Array is great for static operations! `get at(i)` and `set at(i, x)` in `O(1)` time!
- But not so great at dynamic operations...
- (For consistency, we maintain the invariant that array is full)
- Then inserting and removing items requires:
  - reallocating the array
  - shifting all items after the modified item

Sequence Data Structure	**API Type**			**Worst Case $O(\cdot)$**	
Array	Container	Static	Dynamic		
API	`build(x)`	`get_at(i)` `set_at(i)`	`insert_first(x)` `delete_first()`	`insert_last(x)` `delete_last()`	`insert_at(i, x)` `delete_at(i)`
Array	n	1	n	n	n

Linked List Sequence

- Pointer data structure (this is **not** related to a Python "list")
- Each item stored in a **node** which contains a pointer to the next node in sequence
- Each `node` has two fields: `node.item` and `node.next`
- Can manipulate nodes simply by relinking pointers!
- Maintain pointers to the first node in sequence (called the head)
- Can now insert and delete from the front in $\Theta(1)$ time! Yay!
- (Inserting/deleting efficiently from back is also possible; you will do this in PS1)
- But now `get_at(i)` and `set_at(i, x)` each take $O(n)$ time... :(
- Can we get the best of both worlds? Yes! (Kind of...)

Sequence Data Structure	API Type			Worst Case $O(\cdot)$	
Array	Container	Static	Dynamic		
API	`build(x)`	`get_at(i)` `set_at(i)`	`insert_first(x)` `delete_first()`	`insert_last(x)` `delete_last()`	`insert_at(i, x)` `delete_at(i)`
Linked List	n	n	1	n # 1 if we keep track of tail	n

Dynamic Array Sequence

- Make an array efficient for **last** dynamic operations
- Python "list" is a dynamic array
- **Idea!** Allocate extra space so reallocation does not occur with every dynamic operation
- **Fill ratio**: $0 \leq r \leq 1$ the ratio of items to space
- Whenever array is full ($r = 1$), allocate $\Theta(n)$ extra space at end to fill ratio r_i (e.g., 1/2)
- Will have to insert $\Theta(n)$ items before the next reallocation
- A single operation can take $\Theta(n)$ time for reallocation
- However, any sequence of $\Theta(n)$ operations takes $\Theta(n)$ time
- So each operation takes $\Theta(1)$ time "on average"

Sequence Data Structure	API Type			Worst Case $O(\cdot)$	
Array	Container	Static	Dynamic		
API	`build(x)`	`get_at(i)` `set_at(i)`	`insert_first(x)` `delete_first()`	`insert_last(x)` `delete_last()`	`insert_at(i, x)` `delete_at(i)`
Dynamic Array	n	1	n	$1_{(a)}$	n

Amortized Analysis

- Data structure analysis technique to distribute cost over many operations
- Operation has **amortized cost** $T(n)$ if k operations cost at most $\leq kT(n)$
- "$T(n)$ amortized" roughly means $T(n)$ "on average" over many operations
- Inserting into a dynamic array takes $\Theta(1)$ amortized time

Dynamic Array Deletion

- Delete from back? $\Theta(1)$ time without effort, yay!
- However, can be very wasteful in space. Want size of data structure to stay $\Theta(n)$
- **Attempt**: if very empty, resize to r = 1. Alternating insertion and deletion could be bad…
- **Idea!** When $r < r_d$, resize array to ratio r_i where $r_d < r_i$ (e.g., $r_d = 1/4, r_i = 1/2$)
- Then $\Theta(n)$ cheap operations must be made before next expensive resize
- Can limit extra space usage to $(1 + \varepsilon)n$ for any $\varepsilon > 0$ (set $r_d = \frac{1}{1+\epsilon}, r_i = \frac{r_d+1}{2}$)
- Dynamic arrays only support dynamic last operations in $\Theta(1)$ time
- Python List append and pop are amortized $O(1)$ time, other operations can be $O(n)$!
- (Inserting/deleting efficiently from front is also possible; you will do this in PS1)

Sequence Data Structure	API Type			Worst Case $O(\cdot)$	
Array	Container	Static	Dynamic		
API	`build(x)`	`get_at(i)` `set_at(i)`	`insert_first(x)` `delete_first()`	`insert_last(x)` `delete_last()`	`insert_at(i, x)` `delete_at(i)`
Static Array	n	1	n	n	n
Linked List	n	n	1	n # 1 if we keep track of tail	n
Dynamic Array	n	1	n	$1_{(a)}$	n

3 Sorting

Notes

Set Interface

- Storing items in an array in arbitrary order can implement a (not so efficient) set
- Stored items sorted increasing by key allows:
  - faster find min/max (at first and last index of array)
  - faster finds via binary search: $O(logn)$

Set Data Structure	API Type			Worst Case $O(\cdot)$	
Set	Container	Static	Dynamic		
API	`build(x)`	`find(k)`	`insert(k)` `delete(k)`	`find_min()` `find_max()`	`find_prev(k)` `find_next(k)`
Array	n	n	n	n	n
Sorted Array	$nlogn$	$logn$	n	1	$logn$

- But how to construct a sorted array efficiently?

Sorting

- Given a sorted array, we can leverage binary search to make an efficient set data structure.
- **Input**: (static) array A of n numbers
- **Output**: (static) array B which is a sorted permutation of A
  - **Permutation**: array with same elements in a different order
  - **Sorted**: `B[i - 1]` ≤ `B[i]` for all $i \in 1, \ldots, n$
- Example: $[8,2,4,9,3] \rightarrow [2,3,4,8,9]$
- A sort is **destructive** if it overwrites A (instead of making a new array B that is a sorted version of A)
- A sort is **in place** if it uses $O(1)$ extra space (implies destructive: in place ⊆ destructive)

Permutation Sort

- There are $n!$ permutations of A, at least one of which is sorted. (Due to duplications)
- For each permutation, check whether sorted in $\Theta(n)$
- Example: $[2,3,1] \rightarrow [1,2,3],[1,3,2],[2,1,3],[2,3,1],[3,1,2],[3,2,1]$

```python
def permutation_sort(A):
    """Sort A"""
    for B in permutations(A): # O(n!)
        if is_sorted(B): # O(n)
    return B
```

- permutation sort analysis:
  - Correct by case analysis: try all possibilities (Brute Force)
  - Running time: $\Omega(n! \cdot n)$ which is exponential :(

Solving Recurrences

- **Substitution**: Guess a solution, replace with representative function, recurrence holds true
- **Recurrence Tree**: Draw a tree representing the recursive calls and sum computation at nodes
- **Master Theorem**: A formula to solve many recurrences (R03)

Selection Sort

- Find a largest number in prefix `A[:i + 1]` and swap it to `A[i]`
- Recursively sort prefix `A[:i]`
- Example: $[8, 2, 4, 9, 3]$, $[8, 2, 4, 3, 9]$, $[3, 2, 4, 8, 9]$, $[3, 2, 4, 8, 9]$, $[2, 3, 4, 8, 9]$

```python
def selection_sort(A, i=None):
    """Sort A[:i+1]"""
    if i is None: i = len(A) - 1
    if i > 0:
        j = prefix_max(A, i)
        A[i], A[j] = A[j], A[i]
        selection_sort(A, i - 1)

def prefix_max(A, i):
    """Return index of maximum in A[:i+1]"""
    if i > 0:
        j = prefix_max(A, i - 1)
        if A[i] < A[j]:
            return j
    return i
```

- `prefix_max` analysis:
  - Base case: for `i = 0`, array has one element, so index of max is i
  - Induction: assume correct for i, maximum is either the maximum of `A[:i]` or `A[i]`, returns correct index in either case. □
  - $S(1) = \Theta(1); S(n) = S(n-1) + \Theta(1)$
    - Substitution: $S(n) = \Theta(n)$, $cn = \Theta(1) + c(n-1) \implies 1 = \Theta(1)$
    - Recurrence tree: chain of n nodes with $\Theta(1)$ work per node, $\Sigma_{i=0}^{n-1} 1 = \Theta(n)$

Insertion Sort

- Recursively sort prefix `A[:i]`
- Sort prefix `A[:i + 1]` assuming that prefix `A[:i]` is sorted by repeated swaps
- Example: $[8, 2, 4, 9, 3]$, $[2, 8, 4, 9, 3]$, $[2, 4, 8, 9, 3]$, $[2, 4, 8, 9, 3]$, $[2, 3, 4, 8, 9]$

```python
def insertion_sort(A, i=None):
    """Sort A[:i+1]"""
    if i is None: i = len(A) - 1
    if i > 0:
        insertion_sort(A, i-1)
        insert_last(A, i)

def insert_last(A, i):
    """Sort A[:i+1] assuming sorted A[:i]"""
    if i > 0 and A[i] < A[i-1]:
        A[i], A[i-1] = A[i-1], A[i]
        insert_last(A, i-1)
```

- `insert_last` analysis:
  - Base case: for $i = 0$, array has one element so is sorted
  - Induction: assume correct for i, if $A[i] >= A[i-1]$, array is sorted; otherwise, swapping last two elements allows us to sort `A[:i]` by induction. □
  - $S(1) = \Theta(1); S(n) = S(n-1) + \Theta(1) \implies S(n) = \Theta(n)$
- `insertion_sort` analysis:
  - Base case: for $i = 0$, array has one element so is sorted
  - Induction: assume correct for i, algorithm sorts `A[:i]` by induction, and then insert last correctly sorts the rest as proved above. □
  - $T(1) = \Theta(1); T(n) = T(n-1) + \Theta(n) \implies T(n) = \Theta(n^2)$

Merge Sort

- Recursively sort first half and second half (may assume power of two)
- Merge sorted halves into one sorted list (two finger algorithm)
- Example: $[7, 1, 5, 6, 2, 4, 9, 3]$, $[1, 7, 5, 6, 2, 4, 3, 9]$, $[1, 5, 6, 7, 2, 3, 4, 9]$, $[1, 2, 3, 4, 5, 6, 7, 9]$

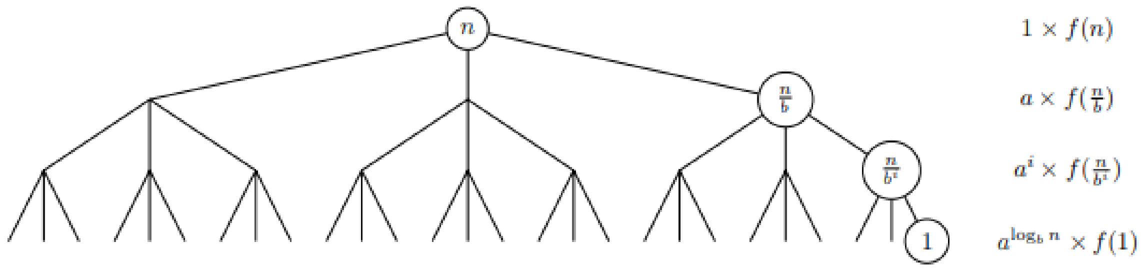
```python
def merge_sort(A, lo=0, hi=None):
    """Sort A[lo:hi]"""
    if hi is None: hi = len(A)
    if hi - lo > 1:
        mid = (lo + hi + 1) // 2
        merge_sort(A, lo, mid)
        merge_sort(A, mid, hi)
        left, right = A[lo:mid], A[mid:hi]
        merge(left, right, A, len(left), len(right), lo, hi)

def merge(left, right, A, i, j, lo, hi):
    """Merge sorted left[:i] anr right[:j] into A[lo:hi]"""
    if lo < hi:
        if (j <= 0) or (i > 0 and left[i-1] > right[j-1]):
            A[hi-1] = left[i-1]
            i -= 1
        else:
            A[hi-1] = right[j-1]
            j -= 1
        merge(left, right, A, i, j, lo, hi -1)
```

- `merge` analysis:
  - Base case: for $n = 0$, arrays are empty, so vacuously correct
  - Induction: assume correct for n, item in `A[r]` must be a largest number from remaining prefixes of `left` and `right`, and since they are sorted, taking largest of last items suffices; remainder is merged by induction. □
  - $S(0) = \Theta(1); S(n) = S(n-1) + \Theta(1) \implies S(n) = \Theta(n)$

- `merge_sort` analysis:
  - Base case: for $n = 1$, array has one element so is sorted
  - Induction: assume correct for $k < n$, algorithm sorts smaller halves by induction, and then merge merges into a sorted array as proved above. $\square$
  - $T(1) = \Theta(1); T(n) = 2T(n/2) + \Theta(n)$
    - Substitution: Guess $T(n) = \Theta(nlogn)$
      $cnlogn = \Theta(n) + 2c(n/2)log(n/2) \implies cnlog(2) = \Theta(n)$
    - Recurrence Tree: complete binary tree with depth $log2n$ and n leaves, level i has 2^i nodes with $O(n/2^i)$ work each, total: $\Sigma_{i=0}^{log_2^n}(2i)(n/2^i) = \Sigma_{i=0}^{log_2^n} n = \Theta(nlogn)$

Master Theorem

- The Master Theorem provides a way to solve recurrence relations in which recursive calls decrease problem size by a constant factor.
- Given a recurrence relation of the form $T(n) = aT(n/b) + f(n)$ and $T(1) = \Theta(1)$, with branching factor $a \geq 1$, problem size reduction factor $b > 1$, and asymptotically non-negative function $f(n)$, the Master Theorem gives the solution to the recurrence by comparing $f(n)$ to $a^{log_b^n} = n^{log_b^a}$, the number of leaves at the bottom of the recursion tree.
- When $f(n)$ grows asymptotically faster than n , the work done at each level decreases geometrically so the work at the root dominates;
- alternatively, when $f(n)$ grows slower, the work done at each level increases geometrically and the work at the leaves dominates.
- When their growth rates are comparable, the work is evenly spread over the tree's $O(logn)$ levels.



case	solution	conditions
1	$T(n) = \Theta(n^{log_b^a})$	$f(n) = \Theta(n^{log_b^{a-\epsilon}})$ for some constant $\varepsilon > 0$
2	$T(n) = \Theta(n^{log_b^a} log^{k+1} n)$	$T(n) = \Theta(n^{log_b^a} log^k n)$ for some constant $k \geq 0$
3	$T(n) = \Theta(f(n))$	$f(n) = \Theta(n^{log_b^{a+\epsilon}})$ for some constant $\varepsilon > 0$ and $af(n/b) < cf(n)$ for some constant $0 < c < 1$

- The Master Theorem takes on a simpler form when f(n) is a polynomial, such that the recurrence has the from $T(n) = aT(n/b) + \Theta(n^c)$ for some constant $c \geq 0$.

case	solution	conditions	intuition
1	$T(n) = \Theta(n^{log_b^a})$	$c < log_b^a$	Work done at leaves dominates
2	$T(n) = \Theta(n^c log^n)$	$c = log_b^a$	Work balanced across the tree
3	$T(n) = \Theta(n^c)$	$c > log_b^a$	Work done at root dominates

- This special case is straight-forward to prove by substitution (this can be done in recitation).
- To apply the Master Theorem (or this simpler special case), you should state which case applies, and show that your recurrence relation satisfies all conditions required by the relevant case.

- There are even stronger (more general) formulas to solve recurrences, but we will not use them in this class.

4 Hashing

Notes

Comparison Model

- In this model, assume algorithm can only differentiate items via comparisons
- **Comparable items**: black boxes only supporting comparisons between pairs
- Comparisons are $<, \leq, >, \geq, =, \neq$, outputs are binary: True or False
- Goal: Store a set of n comparable items, support find(k) operation
- Running time is **lower bounded** by # comparisons performed, so count comparisons!

Decision Tree

- Any algorithm can be viewed as a **decision tree** of operations performed
- An internal node represents a **binary comparison**, branching either True or False
- For a comparison algorithm, the decision tree is binary (draw example)
- A leaf represents algorithm termination, resulting in an algorithm **output**
- A **root-to-leaf** path represents an **execution of the algorithm** on some input
- Need at least one leaf for each **algorithm output**, so search requires $\geq n+1$ leaves

Comparison Search Lower Bound

- What is worst-case running time of a comparison search algorithm?
- running time $\geq$ # comparisons $\geq$ max length of any root-to-leaf path $\geq$ height of tree
- What is minimum height of any binary tree on $\geq n$ nodes?
- Minimum height when binary tree is complete (all rows full except last)
- $Height \geq \lceil lg(n+1) \rceil - 1 = \Omega(logn)$, so running time of any comparison sort is $\Omega(logn)$S
- Sorted arrays achieve this bound! Yay!
- More generally, height of tree with $\Theta(n)$ leaves and max branching factor b is $\Omega(logbn)$
- To get faster, need an operation that allows super-constant $\omega(1)$ branching factor. How??

Direct Access Array

- Exploit Word-RAM $O(1)$ time random access indexing! Linear branching factor!
- **Idea!** Give item unique integer key k in $\{0, \ldots, u-1\}$, store item in an array at index k
- Associate a meaning with each index of array.
- If keys fit in a machine word, i.e. $u \leq 2^w$, worst-case $O(1)$ find/dynamic operations! Yay!
- 6.006: assume input numbers/strings fit in a word, unless length explicitly parameterized
- Anything in computer memory is a binary integer, or use (static) 64-bit address in memory
- But space $O(u)$, so really bad if $n \ll u$... :(
- **Example**: if keys are ten-letter names, for one bit per name, requires $26^{10} \approx 17.6$ TB space
- How can we use less space?

Hashing

- **Idea!** If $n \ll u$, map keys to a smaller range $m = \Theta(n)$ and use smaller direct access array
- **Hash function**: $h(k) : \{0, \ldots, u-1\} \to \{0, \ldots, m-1\}$ (also hash map)
- Direct access array called **hash table**, $h(k)$ called the hash of key k
- If $m \ll u$, no hash function is injective by pigeonhole principle
- Always exists keys a, b such that $h(a) = h(b) \to$ Collision! :(
- Can't store both items at same index, so where to store? Either:

- store somewhere else in the array (**open addressing**)
  - complicated analysis, but common and practical
- store in another data structure supporting dynamic set interface (**chaining**)

Chaining

- **Idea**! Store collisions in another data structure (a chain)
- If keys roughly evenly distributed over indices, chain size is $n/m = n/\Omega(n) = O(1)$!
- If chain has $O(1)$ size, all operations take $O(1)$ time! Yay!
- If not, many items may map to same location, e.g. $h(k) = constant$, chain size is $\Theta(n)$:(
- Need good hash function! So what's a good hash function?

Hash Functions

Division (bad): $h(k) = k \mod m$

- Heuristic, good when keys are uniformly distributed!
- m should avoid symmetries of the stored keys
- Large primes far from powers of 2 and 10 can be reasonable
- Python uses a version of this with some additional mixing
- If $u \ll n$, every hash function will have some input set that will a create $O(n)$ size chain
- **Idea**! Don't use a fixed hash function! Choose one randomly (but carefully)!

Universal (good, theoretically): $h_{ab}(k) = (((ak + b) \mod p) \mod m)$

- Hash Family $\mathcal{H}(p, m) = \{h_{ab} | a, b \in \{0, \dots, p-1\} and a \neq 0\}$
- Parameterized by a fixed prime $p > u$, with a and b chosen from range $\{0, \dots, p-1\}$
- $\mathcal{H}$ is a **Universal** family: $\underset{h \in \mathcal{H}}{Pr}\{h(k_i) = h(k_j)\} \leq 1/m \qquad \forall k_i \neq k_j \in \{0, \dots, u-1\}$
- Why is universality useful? Implies short chain lengths! (in expectation)
- X_{ij} indicator random variable over $h \in \mathcal{H}:\ X_{ij} = 1\ if\ h(k_i) = h(k_j), X_{ij} = 0\ otherwise$
- Size of chain at index $h(k_i)$ is random variable $X_i = \Sigma_j X_{ij}$
- Expected size of chain at index $h(k_i)$:

$$\begin{aligned}
\underset{h \in \mathcal{H}}{\mathbb{E}} X_i = \underset{h \in \mathcal{H}}{\mathbb{E}}\{\Sigma_j X_{ij}\} = \Sigma_j \underset{h \in \mathcal{H}}{\mathbb{E}} X_{ij} &= 1 + \Sigma_{j \neq i} \underset{h \in \mathcal{H}}{\mathbb{E}} X_{ij} \\
&= 1 + \Sigma_{j \neq i}(1) \underset{h \in \mathcal{H}}{Pr}\{h(k_i) = h(k_j)\} + (0) \underset{h \in \mathcal{H}}{Pr}\{h(k_i) \neq h(k_j)\} \\
&\leq 1 + \Sigma_{j \neq i} 1/m = 1 + (n-1)/m
\end{aligned}$$

- Since $m = \Omega(n)$, load factor $\alpha = n/m = O(1)$, so $O(1)$ in expectation!

Dynamic

- If n/m far from 1, rebuild with new randomly chosen hash function for new size m
- Same analysis as dynamic arrays, cost can be **amortized** over many dynamic operations
- So a hash table can implement dynamic set operations in expected amortized O(1) time! :)

Data Structure	API Type			Worst Case $O(\cdot)$	
Set	Container	Static	Dynamic		
API	`build(x)`	`find(k)`	`insert(k)` `delete(k)`	`find_min()` `find_max()`	`find_prev(k)` `find_next(k)`
Array	n	n	n	n	n
Sorted Array	$nlogn$	$logn$	n	1	$logn$
Direct Access Array	u	1	1	u	u
Hash Table	$n_{(e)}$	1_e	$1_{(a)(e)}$	n	n

5 Linear Sorting

Notes

Comparison Sort Lower Bound

- Comparison model implies that algorithm decision tree is binary (constant branching factor)
- Requires # leaves L ≥ # possible outputs
- Tree height lower bounded by $\Omega(logL)$, so worst-case running time is $\Omega(logL)$
- To sort array of n elements, # outputs is n! permutations
- Thus height lower bounded by $log(n!) \geq log((n/2)^{n/2}) = \Omega(nlogn)$
- So merge sort is optimal in comparison model
- Can we exploit a direct access array to sort faster?

Direct Access Array Sort

- Example: $[5, 2, 7, 0, 4]$
- Suppose all keys are unique non-negative integers in range $\{0, \dots, u-1\}$, so $n \leq u$
- Insert each item into a direct access array with size u in $\Theta(n)$
- Return items in order they appear in direct access array in $\Theta(u)$
- Running time is $\Theta(u)$, which is $\Theta(n)$ if $u = \Theta(n)$. Yay!

```
def direct_access_sort(A):
    """Sort A assuming items have distinct non-negative keys."""
    u = 1 + max([x.key for x in A])
    D = [None] * u
    for x in A:
        D[x.key] = x
    i = 0
    for key in range(u):
        if D[key] is not None:
            A[i] = D[key]
            i += 1
```

- What if keys are in larger range, like $u = \Omega(n^2) < n^2$?
- **Idea**! Represent each key k by tuple (a, b) where $k = an + b$ and $0 \leq b < n$
- Specifically $a = \lfloor k/n \rfloor < n$ and $b = (k \mod n)$ (just a 2-digit base-n number!)
- This is a built-in Python operation $(a, b) = divmod(k, n)$
- Example: $[17, 3, 24, 22, 12] \Rightarrow [(3, 2), (0, 3), (4, 4), (4, 2), (2, 2)] \Rightarrow [32, 03, 44, 42, 22]_{(n=5)}$
- How can we sort tuples?

Tuple Sort

- Item keys are tuples of equal length, i.e. item $x.key = (x.k_1, x.k_2, x.k_3, \ldots)$.
- Want to sort on all entries **lexicographically**, so first key k_1 is most significant
- How to sort? **Idea**! Use other **auxiliary sorting algorithms** to separately sort each key
- (Like sorting rows in a spreadsheet by multiple columns)
- What order to sort them in? Least significant to most significant!
- **Exercise**: $[32, 03, 44, 42, 22] \implies [42, 22, 32, 03, 44] \implies [03, 22, 32, 42, 44]_{(n=5)}$
- **Idea**! Use tuple sort with **auxiliary direct access array sort** to sort tuples (a, b).
- **Problem**! Many integers could have the same a or b value, even if input keys distinct
- Need sort allowing **repeated keys** which preserves input order
- Want sort to be **stable**: repeated keys appear in output in same order as input
- Direct access array sort cannot even sort arrays having repeated keys!
- Can we modify direct access array sort to admit multiple keys in a way that is stable?

Counting Sort

- Instead of storing a single item at each array index, store a chain, just like hashing!
- For stability, chain data structure should remember the order in which items were added
- Use a **sequence** data structure which maintains insertion order
- To insert item `x`, `insert_last` to end of the chain at index $x.key$
- Then to sort, read through all chains in sequence order, returning items one by one

```
def counting_sort(A):
    """Sort A assuming items have non-negative keys."""
    u = 1 + max([x.key for x in A])
    D = [[] for i in range(u)]
    for x in A:
        D[x.key].append(x)
    i = 0
    for chain in D:
        for x in chain:
            A[i] = x
            i += 1
```

Radix Sort

- **Idea**! If $u < n^2$, use tuple sort with **auxiliary counting sort to sort** tuples (a, b)
- Sort least significant key b, then most significant key a
- Stability ensures previous sorts stay sorted
- Running time for this algorithm is $O(2n) = O(n)$. Yay!
- If every $key < n^c$ for some positive $c = logn(u)$, every key has at most c digits base n
- A c-digit number can be written as a c-element tuple in $O(c)$ time
- We sort each of the c base-n digits in $O(n)$ time
- So tuple sort with auxiliary counting sort runs in $O(cn)$ time in total
- If c is constant, so each key is $\leq n^c$, this sort is linear $O(n)$!

```
def radix_sort(A):
    """Sort A assuming items have non-negative keys"""
    n = len(A)
    u = 1 + max([x.key for x in A])
    c = 1 + (u.bit_length() // n.bit_length())

    class Obj: pass

    D = [Obj() for a in A]
    for i in range(n):
```

```
        D[i].digits = []
        D[i].item = A[i]
        high = A[i].key
        for j in range(c):
            high, low = divmod(high, n)
            D[i].digits.append(low)
    for i in range(c):
        for j in range(n):
            D[j].key = D[j].digits[i]
        counting_sort(D)
    for i in range(n);
        A[i] = D[i].item
```

Algorithm	Time $O(\cdot)$	In-place?	Stable?	Comments
Insertion Sort	n^2	Y	Y	$O(nk)$ for k-proximate
Selection Sort	n^2	Y	N	$O(n)$ swaps
Merge Sort	$nlogn$	N	Y	stable, optimal comparison
Counting Sort	$n+u$	N	Y	$O(n)$ when $u=O(n)$
Radix Sort	$n+nlog_n u$	N	Y	$O(n)$ when $u=O(n)$

6 Binary Trees, Part 1

Notes

Sequence Data Structure	API Type			Worst Case $O(\cdot)$	
Array	Container	Static	Dynamic		
API	`build(x)`	`get_at(i)` `set_at(i)`	`insert_first(x)` `delete_first()`	`insert_last(x)` `delete_last()`	`insert_at(i, x)` `delete_at(i)`
Static Array	n	1	n	n	n
Linked List	n	n	1	n # 1 if we keep track of tail	n
Dynamic Array	n	1	n	$1_{(a)}$	n
Goal	n	$logn$	$logn$	$logn$	$logn$

Set Data Structure	API Type			Worst Case $O(\cdot)$	
Set	Container	Static	Dynamic		
API	`build(x)`	`find(k)`	`insert(k)` `delete(k)`	`find_min()` `find_max()`	`find_prev(k)` `find_next(k)`
Array	n	n	n	n	n
Sorted Array	$nlogn$	$logn$	n	1	$logn$
Goal	$nlogn$	$logn$	$logn$	$logn$	$logn$

How? Binary Trees!

- Pointer-based data structures (like Linked List) can achieve **worst-case** performance
- Binary tree is pointer-based data structure with three pointers per node
- Node representation: `node.{item, parent, left, right}`
- Example:

```
             _________<A>_____         node   | <A> | <B> | <C> | <D> | <E> | <F> |
        __<B>_____          <C>        item   |  A  |  B  |  C  |  D  |  E  |  F  |
    __<D>       <E>                    parent |  -  | <A> | <A> | <B> | <B> | <D> |
  <F>                                  left   | <B> | <C> |  -  | <F> |  -  |  -  |
                                       right  | <C> | <D> |  -  |  -  |  -  |  -  |
```

```python
class TreeNode:
    def __init__(self, x):
        self.item = x
        self.left = None
        self.right = None
        self.parent = None
```

Terminology

- The root of a tree has no parent (Ex: `<A>`)
- leaf of a tree has no children (Ex: `<C>`, `<E>`, and `<F>`)
- Define **`depth(<X>)`** of node `<X>` in a tree rooted at `<A>` to be length of path from `<A>` to `<X>`
- Define **`height(<X>)`** of node `<X>` to be max depth of any node in the **subtree** rooted at `<X>`
- **Idea**: Design operations to run in $O(h)$ time for root height h, and maintain $h = O(logn)$
- A binary tree has an inherent order: its **traversal order** (**In-order traversal**)
  - every node in node `<X>`'s left subtree is before `<X>`
  - every node in node `<X>`'s right subtree is after `<X>`

```python
def subtree_iter(A):
    if A.left: yield from A.left.subtree_iter()
    yield A
    if A.right: yield from A.right.subtree_iter()
```

- List nodes in traversal order via a recursive algorithm starting at root:
  - Recursively list left subtree, list self, then recursively list right subtree
  - Runs in $O(n)$ time, since $O(1)$ work is done to list each node
  - **Example**: Traversal order is (`<F>`, `<D>`, `<B>`, `<E>`, `<A>`, `<C>`)
- Right now, traversal order has no meaning relative to the stored items
- Later, assign semantic meaning to traversal order to implement Sequence/Set interfaces

Tree Navigation

- **Find first** node in the traversal order of node `<X>`'s subtree (last is symmetric)
  - Otherwise, `<X>` is the first node, so return it
  - Running time is $O(h)$ where h is the height of the tree
  - **Example**: first node in `<A>`'s subtree is `<F>`

```python
def subtree_first(A):
    if A.left: return A.left.subtree_first()
    return A

def subtree_last(A):
    if A.right: return A.right.subtree_last()
    return A
```

- **Find successor** of node `<X>` in the traversal order (predecessor is symmetric)
    - If `<X>` has right child, return first of right subtree
    - Otherwise, return lowest ancestor of `<X>` for which `<X>` is in its left subtree
    - Running time is $O(h)$ where h is the height of the tree
    - **Example**: Successor of: `<B>` is `<E>`, `<E>` is `<A>`, and `<C>` is None.

```python
def successor(A):
    if A.right: return A.right.subtree_first()
    while A.parent and (A is A.parent.right):
        A = A.parent
    return A.parent

def predecessor(A):
    if A.left: return A.left.subtree_last()
    while A.parent and (A is A.parent.left):
        A = A.parent
    return A.parent
```

Dynamic Operations

- Change the tree by a single item (only add or remove leaves):
    - add a node after another in the traversal order (before is symmetric)
    - remove an item from the tree
- **Insert** node `<Y>` after `<X>` in the traversal order
    - If `<X>` has no right child, make `<Y>` the right child of `<X>`
    - Otherwise, make `<Y>` the left child of `<X>`'s successor (which cannot have a left child)
    - Running time is $O(h)$ where h is the height of the tree
        - **Example:** Insert node `<G>` before `<E>` in traversal order

```
        ______<A>__                     _________<A>__
     __<B>__     <C>    =>           __<B>______     <C>
  __<D>    <E>                    __<D>      __<E>
 <F>                             <F>        <G>
```

- **Example:** Insert node `<H>` after `<A>` in traversal order

```
         _________<A>____                   _________<A>______
      __<B>______      <C>    =>         __<B>______       __<C>
   __<D>      __<E>                   __<D>      __<E>    <H>
  <F>        <G>                     <F>        <G>
```

```
def subtree_insert_before(A, B):
    if A.left:
        A = A.left.subtree_last()
        A.right, B.parent = B, A
    else:
        A.left, B.parent = B, A

def subtree_insert_after(A, B):
    if A.right:
        A = A.right.subtree_first()
        A.left, B.parent = B, A
    else:
        A.right, B.parent = B, A
```

- **Delete** the item in node `<X>` from `<X>`'s subtree
  - If `<X>` is a leaf, detach from parent and return
  - Otherwise, `<X>` has a child
    - If `<X>` has a left child, swap items with the predecessor of `<X>` and recurse
    - Otherwise `<X>` has a right child, swap items with the successor of `<X>` and recurse
  - Running time is $O(h)$ where h is the height of the tree

  – **Example:** Remove `<F>` (a leaf)

```
          _________<A>_____              _________<A>_____
      __<B>_____        __<C>    =>   __<B>_____        __<C>
   __<D>      __<E>    <H>           <D>      __<E>    <H>
  <F>        <G>                             <G>
```

  – **Example:** Remove `<A>` (not a leaf, so first swap down to a leaf)

```
      _________<A>_____              _________<E>_____             ______<E>_____
   __<B>_____        __<C>   =>   __<B>_____        __<C>   =>   __<B>__      __<C>
  <D>      __<E>    <H>          <D>      __<G>    <H>          <D>    <G>    <H>
         <G>                            <A>
```

```
def subtree_delete(A):
    if A.left or A.right:    # A is a leaf node
        if A.left: B = A.predecessor()
        else:      B = A.successor()
        A.item, B.item = B.item, A.item
    if A.parent:             # A is not a leaf node
        if A.parent.left is A: A.parent.left = None
        else:                  A.parent.right = None
    return A
```

Application: Set

- **Idea**! **Set Binary Tree** (a.k.a Binary Search Tree / BST)
- Traversal order(In-order) is sorted order increasing by key
  - Equivalent to **BST Property**: for every node, every key in left subtree $\leq$ node's key $\leq$ every key in right subtree
- Then can find the node with key k in node `<X>`'s subtree in $O(h)$ time like binary search:
  - If k is smaller than the key at `<X>`, recurse in left subtree (or return None)
  - If k is larger than the key at `<X>`, recurse in right subtree (or return None)
  - Otherwise, return the item stored at `<X>`
- Other Set operations follow a similar pattern

```python
class BSTNode(TreeNode):
    def subtree_find(A, k):
        if k == A.item.key: return A
        if k < A.item.key and A.left: return A.left.subtree_find(k)
        if k > A.item.key and A.right: return A.right.subtree_find(k)

    def subtree_find_next(A, k):
        if A.item.key <= k:
            if A.right: return A.right.subtree_find_next(k)
            else:       return None
        if A.item.key > k:
            if A.left:
                B = A.left.subtree_find_next(k)
                if B: return B
        return A

    def subtree_find_prev(A, k):
        if A.item.key >= k:
            if A.left: return A.left.subtree_find_prev(k)
            else:      return None

        if A.item.key < k:
            if A.right:
                B = A.right.subtree_find_prev(k)
                if B: return B
        return A

    def subtree_insert(A, B):
        if B.item.key < A.item.key:
            if A.left: A.left.subtree_insert(B)
            else:      A.subtree_insert_before(B)
        elif B.item.key > A.item.key:
            if A.right: A.right.subtree_insert(B)
            else:       A.subtree_insert_after(B)
        else: A.item = B.item

class BinaryTree:
    def __init__(self, node_type=BinaryNode):
        self.root = None
        self.size = 0
        self.node_type = node_type

    def __len__(self): return self.size
    def __iter__(self):
        if self.root:
            for item in self.root.subtree_iter():
                yield node.item

class BinaryTreeSet(BinaryTree):
    def __init__(self):
        super().__init__(node_type=BSTNode)

    def iter_order(self): yield from self

    def build(self, X):
        for x in X: self.insert(x)

    def find_min(self):
        if self.root: return self.root.subtree_first().item

    def find_max(self):
        if self.root: return self.root_subtree_last().item
```

```python
    def find(self, k):
        if self.root:
            node = self.root.subtree_find(k)
            if node: return node.item

    def find_next(self, k):
        if self.root:
            node = self.root.subtree_find_next(k)
            if node: return node.item

    def find_prev(self, k):
        if self.root:
            node = self.root.subtree_find_prev(k)
            if node: return node.item

    def insert(self, x):
        new_node = self.node_type(x)
        if self.root:
            self.root.subtree_insert(new_node)
            if new_node.parent is None: return False
        else:
            self.root = new_node
        self.size += 1

    def delete(self, k):
        assert self.root
        node = self.root.subtree_find(k)
        assert node
        ext = node.subtree_delete()
        if ext.parent is None: self.root = None
        self.size -= 1
        return ext.item
```

Application: Sequence

- **Idea**! **Sequence Binary Tree**: Traversal order is sequence order
- How do we find i^th node in traversal order of a subtree? Call this operation `subtree_at(i)`
- Could just iterate through entire traversal order, but that's bad, $O(n)$
- However, if we could compute a subtree's **size** in $O(1)$, then can solve in $O(h)$ time
  - How? Check the size n_L of the left subtree and compare to i
  - If $i < n_L$, recurse on the left subtree
  - If $i > n_L$, recurse on the right subtree with $i' = i - n_L - 1$
  - Otherwise, $i = n_L$, and you've reached the desired node!
- Maintain the size of each node's subtree at the node via **augmentation**
  - Add `node.size` field to each `node`
  - When adding new leaf, add $+1$ to `a.size` for all ancestors a in $O(h)$ time
  - When deleting a leaf, add -1 to `a.size` for all ancestors a in $O(h)$ time
- Sequence operations follow directly from a fast `subtree_at(i)` operation
- Naively, `build(X)` takes $O(nh)$ time, but can be done in $O(n)$ time; see recitation

7 Binary Tree II: AVL

Notes

Height Balance

- How to maintain height $h = O(log n)$ where n is number of nodes in tree?
- A binary tree that maintains $O(log n)$ height under dynamic operations is called **balanced**
  - There are many balancing schemes (Red-Black Trees, Splay Trees, 2-3 Trees, ...)
  - First proposed balancing scheme was the **AVL Tree**(Adelson-Velsky and Landis, 1962)

Rotations

- Need to reduce height of tree without changing its traversal order, so that we represent the same sequence of items.
- How to change the structure of a tree, while preserving traversal order? **Rotations**!

```
        ______<D>__      rotate_right(<D>)      __<B>______
    ___<B>__    <E>             =>             <A>     __<D>__
   <A>    <C>   / \                            / \    <C>    <E>
   / \    / \  /___\            <=            /___\   / \    / \
  /___\  /___\            rotate_left(<B>)           /___\  /___\
```

- A rotation relinks $O(1)$ pointers to modify tree structure and maintains traversal order

```python
def subtree_rotate_right(D):
    assert D.left
    B, E = D.left, D.right
    A, C = B.left, B.right

    # make sure new B has the right connection to D's parent
    D, B = B, D
    D.item, B.item = D.item, B.item

    B.left, B.right = A, D
    D.left, D.right = C, E

    if A: A.parent = B
    if E: E.parent = D

def subtree_rotate_left(B):
    assert B.right
    A, D = B.left, B.right
    C, E = D.left, D.right

    B, D = D, B
    B.item, D.item = D.item, B.item

    D.left, D.right = B, E
    B.left, B.right = A, C

    if A: A.parent = B
    if E: E.parent = D
```

Rotations Suffice

- **Claim**: $O(n)$ rotations can transform a binary tree to any other with same traversal order
- **Proof**: Repeatedly perform last possible right rotation in traversal order; resulting tree is a canonical chain. Each rotation increases depth of the last node by 1. Depth of last node in final chain is $n - 1$, so at most $n - 1$ rotations are performed. Reverse canonical rotations to reach target tree. Q.E.D
- Can maintain height-balance by using $O(n)$ rotations to fully balance the tree, but slow :(
- We will keep the tree balanced in $O(log n)$ time per operation!

AVL Trees: Height Balance

- AVL trees maintain **height-balance** (also called the **AVL property**)
  - A node is **height-balanced** if heights of its left and right subtree differ by at most 1
  - Let **skew** of a node be the height of its right subtree minus that of its left subtree
  - Then a node is height-balanced if its skew is $-1, 0$ or 1
- **Claim**: A binary tree with height-balanced nodes has height $h = O(logn)$ (i.e., $n = 2^{\Omega(h)}$)
- **Proof**: Suffices to show fewest nodes $F(h)$ in any height h tree is $F(h) = 2^{\Omega(h)}$

$$F(0) = 1, F(1) = 2, F(h) = 1 + F(h-1) + F(h-2) \geq 2F(h-2)) \implies F(h) \geq 2^{h/2} \qquad \square$$

- Suppose adding or removing leaf from a height-balanced tree results in imbalance
  - Only subtree of the leaf's ancestors have changed in height or skew
  - Heights changed by only ± 1, so skews still have magnitude ≤ 2
  - **Idea**: Fix height-balance of ancestors starting from leaf up to the root
  - Repeatedly rebalanced lowest ancestor that is not height-balanced, wlog assume skew 2
- Local Rebalance: Given binary tree node `<B>`:
  - whose skew 2 and
  - every other node in `<B>`'s subtree is height-balanced
  - then `<B>`'s subtree can be made height-balanced via one or two rotations
  - (after which `<B?`'s height is the same or one less than before)
- **Proof**:
  - Since skew of `<B>` is 2, `<B>?`'s right child exists
  - **Case 1**: skew of `<F>` is 0 or **Case 2**: skew of `<F>` is 1
    - Perform a left rotation on `<B>`

TBC

Computing Height

- How to tell whether node is height-balanced? Compute heights of subtrees!
- How to compute the height of node `<X>` ? Naive algorithm:
  - Recursively compute height of the left and right subtrees of `<X>`
  - Add 1 to the max of the two heights
  - Runs in $\Omega(n)$ time, since we recurse on every node :(
- **Idea**: Augment each node with the height of its subtree! (Save for later!)
- Height of `<X>` can be computed in $O(1)$ time from the heights of its children:
  - Look up the stored heights of left and right subtrees in $O(1)$ time
  - Add 1 to the max of the two heights
- During dynamic operations, we must **maintain** our augmentation as the tree changes shape
- Recompute subtree augmentations at every node whose subtree changes:
  - Update relinked nodes in a rotation operation in $O(1)$ time (ancestors don't change)
  - Update all ancestors of an inserted or deleted node in $O(h)$ time by walking up the tree

Steps to Augment a Binary Tree

- In general, to augment a binary tree with a subtree property P, you must:
  - State the subtree property `P(<X>)` you want to store at each node `<X>`
  - Show how to compute `P(<X>)` from the augmentations of `<X>`'s children in $O(1)$ time
  - Then stored property `P(<X>)` can be maintained without changing dynamic operation costs

Application: Sequence

- For sequence binary tree, we needed to know subtree **sizes**
- For just inserting/deleting a leaf, this was easy, but now need to handle rotations
- Subtree size is a subtree property, so can maintain via augmentation
  - Can compute size from sizes of children by summing them and adding 1

Conclusion

- Set AVL trees achieve $O(logn)$ time for all set operations
- except $O(nlogn)$ time for build and $O(n)$ time for iter
- Sequence AVL trees achieve $O(logn)$ time for all sequence operations
- except $O(n)$ time for build and iter

Application: Sorting

- Any Set data structure defines a sorting algorithm: build (or repeatedly insert) then iter
- For example, Direct Access Array Sort from Lecture 5
- **AVL** Sort is a new $O(nlogn)$ time sorting algorithm

8 Binary Heaps

Notes

Priority Queue Interface

- Keep track of many items, quickly access/remove the most important
  - Example: router with limited bandwidth, must prioritize certain kinds of messages
  - Example: process scheduling in operating system kernels
  - Example: discrete-event simulation (when is next occurring event?)
  - Example: graph algorithms (later in the course)
- Order items by key = priority so **Set interface** (not Sequence interface)
- Optimized for a particular subset of Set operations:

Operation	Specification
`build(X)`	build priority queue from iterable X
`insert(x)`	add item x to data structure
`delete_max()`	remove and return stored item with largest key
`find_max()`	return stored item with largest key

- (Usually optimized for max or min, not both)
- Focus on `insert` and `delete_max` operations: `build` can repeatedly `insert`; `find_max()` can `insert(delete_min())`

```
class PriorityQueue:
    def __init__(self):
        self.A = []

    def insert(self, x):
        self.A.append(x)

    def delete_max(self):
```

```
        assert len(self.A) > 0
        return self.A.pop()   # not correct by it self.

    @classmethod
    def sort(PQ, A):
        pq = PQ()
        for x in A: pq.insert(x)
        out = [pq.delete_max() for _ in A]
        return reversed(out)
```

Priority Queue Sort

- Any priority queue data structure translates into a sorting algorithm:
  - `build(A)`, e.g., insert items one by one in input order
  - Repeatedly `delete_min()` (or `delete_max()`) to determine (reverse) sorted order
- All the hard work happens inside the data structure
- Running time is $T_{build} + n \cdot T_{delete_max} \leq n \cdot T_{insert} + n \cdot T_{delete_max}$
- Many sorting algorithms we've seen can be viewed as priority queue sort:

Priority Queue Data Structure		**Operations** $O(\cdot)$		**Priority Queue Sort**		**Algorithm**
	`build(A)`	`insert(x)`	`delete_max()`	Time	In-place?	
Dynamic Array	n	$1_{(a)}$	n	n^2	Y	Selection Sort
Sorted Dynamic Array	$nlogn$	n	$1_{(a)}$	n^2	Y	Insertion Sort
Set AVL Tree	$nlogn$	$logn$	$logn$	$nlogn$	N	AVL Sort
Goal	n	$logn_{(a)}$	$logn_{(a)}$	$nlogn$	Y	Heap Sort

Priority Queue: Set AVL Tree

- Set AVL trees support `insert(x)`, `find_min()`, `find_max()`, `delete_min()`, and `delete_max()` in $O(logn)$ time per operation
- So priority queue sort runs in $O(nlogn)$ time
  - This is (essentially) AVL sort from Lecture 7
- Can speed up `find_min()` and `find_max()` to $O(1)$ time via subtree augmentation
- But this data structure is complicated and resulting sort is not in-place
- Is there a simpler data structure for just priority queue, and in-place $O(nlgn)$ sort? YES, binary heap and heap sort
- Essentially implement a Set data structure on top of a Sequence data structure (array), using what we learned about binary trees

Priority Queue: Array

- Store elements in an **unordered** dynamic array
- `insert(x)`: append x to end in amortized $O(1)$ time
- `delete_max()`: find max in $O(n)$, swap max to the end an   d remove
- `insert` is quick, but `delete_max` is slow
- Priority queue sort is selection sort! (plus some copying)

```python
class PQArray(PriorityQueue):
    def delete_max(self):  # O(n)
        n, A, m = len(self.A), self.A, 0
        for i in range(1, n):
            m = i if A[m].key < A[i].key else m
        A[m], A[n] = A[n], A[m]
        return super().delete_max()  # pop from end
```

We use `*args` to allow insert to take one argument (as makes sense now) or zero arguments; we will need the latter functionality when making the priority queues in-place.

Priority Queue: Sorted Array

- Store elements in a sorted dynamic array
- `insert(x)`: append `x` to end, swap down to sorted position in $O(n)$ time
- `delete_max()`: delete from end in $O(1)$ amortized
- `delete_max` is quick, but `insert` is slow
- Priority queue sort is insertion sort! (plus some copying)
- Can we find a compromise between these two array priority queue extremes?

```python
class PQSortedArray(PriorityQueue):
    def insert(self, x=None):
        if x is not None: super().insert(x)
        i, A = len(self.A) - 1, self.A
        while 0 < i and A[i+1].key < A[i].key:
            A[i+1], A[i] = A[i], A[i+1]
            i -= 1
```

Array as a Complete Binary Tree

- **Idea**: interpret an array as a complete binary tree, with maximum 2^i nodes at depth i except at the largest depth, where all nodes are **left-aligned**
- Equivalently, complete tree is filled densely in reading order: root to leaves, left to right
- Perspective: **bijection** between arrays and complete binary trees
- Height of complete tree perspective of array of n item is $\lceil logn \rceil$, so balanced binary tree

```
d0          ________0_____
d1      _____0_____    __0__
d2    __0__      __0  0    0
d3    0    0    0
```

```
Q = [0,1,2,3,4,5,6,7,8,9]
d0   0                         ->          ________0_____
d1     1 2                     ->      _____1_____    __2__
d2         3 4 5 6             ->    __3__      __4  5    6
d3                 7 8 9       ->    7    8    9
```

Implicit Complete Tree

- Complete binary tree structure can be implicit instead of storing pointers
- Root is at index 0
- Compute neighbors by index arithmetic:

$$left(i) = 2i + 1$$
$$right(i) = 2i + 2$$
$$parent(i) = \lfloor \frac{i-1}{2} \rfloor$$

Binary Heaps

- **Idea**: keep larger elements higher in tree, but only locally
- **Max-Heap Property** at node $i : Q[i] \geq Q[j] for j \in \{left(i), right(i)\}$
- **Max-heap** is an array satisfying max-heap property at all nodes
- **Claim**: In a max-heap, every node i satisfies $Q[i] \geq Q[j]$ for all nodes j in `subtree(i)`
- **Proof**:
  - Induction on $d = depth(j) - depth(i)$
  - Base case: $d = 0$ implies $i = j$ implies $Q[i] \geq Q[j]$ (in fact, equal)
  - $depth(parent(j)) - depth(i) = d - 1 < d$, so $Q[i] \geq Q[parent(j)]$ by induction
  - $Q[parent(j)] \geq Q[j]$ by Max-Heap Property at `parent(j)`
- In particular, max item is at root of max-heap

```
def parent(i):
    p = (i - 1) // 2
    return p if 0 < i else i

def left(i, n):
    l = 2 * i + 1
    return l if l < n else i

def right(i, n):
    r = 2 * i + 2
    return r if r < n else i
```

Heap Insert

- Append new item x to end of array in $O(1)$ amortized, making it next leaf i in reading order
- `max_heapify_up(i)`: swap with parent until Max-Heap Property
  - Check whether $Q[parent(i)] \geq Q[i]$ (part of Max-Heap Property at parent(i))
  - If not, swap items $Q[i]$ and $Q[parent(i)]$, and recursively `max_heapify_up(parent(i))`
- Correctness:
  - Max-Heap Property guarantees all nodes $\geq$ descendants, except $Q[i]$ might be > some of its ancestors (unless i is the root, so we're done)
  - If swap is necessary, same guarantee is true with $Q[parent(i)]$ instead of $Q[i]$
- Running time: height of tree, so $\Theta(\log n)$

Heap Delete Max

- Can only easily remote last element from dynamic array, but max key is in root of tree
- So swap item at root node $i = 0$ with last item at node $n - 1$ in heap array
- `max_heapify_down(i)`: swap root with larger child until Max-Heap Property
  - Check whether $Q[i] \geq Q[j] \ for \ j \in \{left(i), right(i)\}$ (Max-Heap Property at i)
  - If not, swap $Q[i]$ with $Q[j]$ for child $j \ \in \{left(i), right(i)\}$ with maximum key, and recursively `max_heapify_down(j)`
- Correctness:
  - Max-Heap Property guarantees all nodes $\geq$ descendants, except $Q[i]$ might be < some descendants (unless i is a leaf, so we're done)
  - If swap is necessary, same guarantee is true with $Q[j]$ instead of $Q[i]$
- Running time: height of tree, so $\Theta(\log n)$

```
class PQHeap(PriorityQueue):
```

```python
    def insert(self, x=None):
        if x: super().insert(x)
        n, A = self.n, self.A
        max_heapify_up(A, n, n-1)

    def delete_max(self):
        n, A = self.n, self.A
        A[0], A[n] = A[n], A[0]
        max_heapify_down(A, n, 0)
        return super().delete_max()

def max_heapify_up(A, n, c):
    p = parent(c)
    if A[p].key < A[c].key:
        A[c], A[p] = A[p], A[c]
        max_heapify_up(A, n, p)


def max_heapify_down(A, n, p):
    l, r = left(p, n), right(p, n)
    c = l if A[r].key < A[l].key else r
    if A[p].key < A[c].key:
        A[c], A[p] = A[p], A[c]
        max_heapify_down(A, n, c)
```

Heap Sort

- Plugging max-heap into priority queue sort gives us a new sorting algorithm
- Running time is $O(nlogn)$ because each `insert` and `delete_max` takes $O(logn)$
- But often include two improvements to this sorting algorithm:

In-place Priority Queue Sort

- Max-heap Q is a prefix of a larger array A, remember how many items $|Q|$ belong to heap
- $|Q|$ is initially zero, eventually $|A|$ (after inserts), then zero again (after deletes)
- `insert()` absorbs next item in array at index $|Q|$ into heap
- `delete_max()` moves max item to end, then abandons it by decrementing $|Q|$
- In-place priority queue sort with Array is exactly Selection Sort
- In-place priority queue sort with Sorted Array is exactly Insertion Sort
- In-place priority queue sort with binary Max Heap is Heap Sort

```python
class PriorityQueue:
    def __init__(self, A):
        self.n, self.A = 0, A

    def insert(self):
        assert self.n < len(self.A)
        self.n += 1

    def delete_max(self):
        assert self.n >= 1
        self.n -= 1

    @classmethod
    def sort(Queue, A):
        pq = Queue(A)
        for i in range(len(A)): pq.insert()
        for i in range(len(A)): pq.delete_max()
        return pq.A
```

Linear Build Heap

- Inserting n items into heap call `max_heapify_up(i)` for i from 0 to $n-1$ (root down):

$$worst-case\ swaps \approx \Sigma_{i=0}^{n-1} depth(i) = \Sigma_{i=0}^{n-1} logi = log(n!) \geq (n/2)log(n/2) = \Omega(nlogn)$$

- **Idea**! Treat full array as a complete binary tree from start, then `max_heapify_down(i)` for i from $n-1$to 0 (leaves up):

$$worst-case\ swaps \approx \Sigma_{i=0}^{n-1} height(i) = \Sigma_{i=0}^{n-1}(logn - logi) = log(\frac{n^n}{n!}) = \Theta(log(\frac{n^n}{\sqrt{n}(n/e)^n})) = O(n)$$

- So can `build` heap in $O(n)$ time
- (Doesn't speed up $O(nlogn)$ performance of heap sort)

```
def build_max_heap(A):
    n = len(A)
    for i in range(n // 2, -1, -1):
        max_heapify_down(A, n, i)
```

Sequence AVL Tree Priority Queue

- Where else have we seen linear build time for an otherwise logarithmic data structure? Sequence AVL Tree!
- Store items of priority queue in Sequence AVL Tree in **arbitrary order** (insertion order)
- Maintain max (and/or min) augmentation:
  - `node.max` = pointer to node in subtree of `node` with maximum key
    - This is a subtree property, so constant factor overhead to maintain
- `find_min()` and `find _max()` in $O(1)$ time
- `delete_min()` and `delete_max()` in $O(logn)$ time
- `build(A)` in $O(n)$ time
- Same bounds as binary heaps (and more)

Set vs. Multiset

- While our Set interface assumes no duplicate keys, we can use these Sets to implement Multisets that allow items with duplicate keys:
  - Each item in the Set is a Sequence (e.g., linked list) storing the Multiset items with the same key, which is the key of the Sequence
- In fact, without this reduction, binary heaps and AVL trees work directly for duplicate-key items (where e.g. `delete_max` deletes some item of maximum key), taking care to use ≤ constraints (instead of < in Set AVL Trees)

9 Breadth-First Search

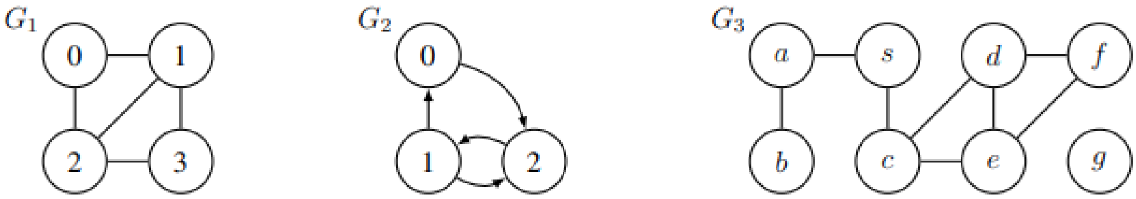
Notes

Graph Applications

- Why? Graphs are everywhere!
- any network system has direct connection to graphs
- e.g., road networks, computer networks, social networks
- the state space of any discrete system can be represented by a transition graph
- e.g., puzzle & games like Chess, Tetris, Rubik's cube

- e.g., application workflows, specifications

Graph Definitions



- Graph $G = (V, E)$ is a set of vertices V and a set of pairs of vertices $E \subseteq V \times X$
- **Directed** edges are ordered pairs, e.g., (u, v) for $u, v \in V$
- **Undirected** edges are unordered pairs, u, v for $u, v \in V$ i.e., (u, v) and (v, u)
- In this class, we assume all graphs are **simple**:
    - **edges are distinct**, e.g., (u, v) only occurs once in **E** (though (v, u) may appear), and
    - edges are **pairs of distinct vertices**, e.g., $u \neq v$ for all $(u, v) \in E$
    - Simple implies $|E| = O(|V|^2)$, since $|E| \leq \binom{|V|}{2}$ for undirected, $\leq 2\binom{|V|}{2}$ for directed

Neighbor Sets/Adjacencies

- The **outgoing neighbor set** of $u \in V$ is $Adj^+(u) = \{v \in V | (u, v) \in E\}$
- The **incoming neighbor set** of $u \in V$ is $Adj^-(u) = \{v \in V | (v, u) \in E\}$
- The **out-degree** of a vertex $u \in V$ is $deg^+(u) = |Adj^+(u)|$
- The **in-degree** of a vertex $u \in V$ is $deg^-(u) = |Adj^-(u)|$
- For undirected graphs, $Adj^-(u) = Adj^+(u)$ and $deg^-(u) = deg^+(u)$
- Dropping superscript defaults to outgoing, i.e., $Adj(u) = Adj^+(u)$ and $deg(u) = deg^+(u)$

Graph Representations

- To store a graph $G = (V, E)$, we need to store the outgoing edges $Adj(u)$ for all $u \in V$
- First, need a Set data structure Adj to map u to $Adj(u)$
- Then for each u, need to store $Adj(u)$ in another data structure called an **adjacency list**
- Common to use **direct access array** or **hash table** for Adj, since want lookup fast by vertex
- Common to use **array** or **linked list** for each $Adj(u)$ since usually only iteration is needed
- For the common representations, Adj has size $\Theta(|V|)$, while each $Adj(u)$ has size $\Theta(deg(u))$
- Since $\Sigma_{u \in V} deg(u) \leq 2|E|$ by handshaking lemma, graph storable in $\Theta(|V| + |E|)$ space
- Thus, for algorithms on graphs, **linear time** will mean $\Theta(|V| + |E|)$ (linear in size of graph)

Paths

- **A path** is a sequence of vertices $p = (v_1, v_2, \ldots, v_k)$ where $(v_i, v_{i+1}) \in E$ for all $1 \leq i < k$.
- A path is **simple** if it does not repeat vertices
- The length $l(p)$ of a path p is the number of edges in the path
- The distance $\delta(u, v)$ from $u \in V$ to $v \in V$ is the minimum length of any path from u to v
    - i.e., the length of a **shortest path** from u to v
    - (by convention, $\delta(u, v) = \infty$ if u is not connected to v)

Graph Path Problems

- There are many problems you might want to solve concerning paths in a graph:
- SINGLE_PAIR_REACHABILITY(G, s, t):
  is there a path in G from $s \in V$ to $t \in V$?

- SINGLE_PAIR_SHORTEST_PATH(G, s, t):
  return distance $\delta(s, t)$, and a shortest path in $G = (V, E)$ from $s \in V$ to $t \in V$
- SINGLE_SOURCE_SHORTEST_PATHS(G, s):
  return $\delta(s, v)$ for all $v \in V$, and a **shortest-path tree** containing a shortest path from s to every $v \in V$ (defined below)
- Each problem above is at least as hard as every problem above it
  (i.e., you can use a black-box that solves a lower problem to solve any higher problem)
- We won't show algorithms to solve all of these problems
- Instead, show one algorithm that solves the hardest in $O(|V| + |E|)$ time!

Shortest Paths Tree

- How to return a shortest path from source vertex s for every vertex in graph?
- Many paths could have length $\Omega(|V|)$, so returning every path could require $\Omega(|V|^2)$ time
- Instead, for all $v \in V$, store its **parent** $P(v)$: second to last vertex on a shortest path from s
- Let $P(s)$ be null (no second to last vertex on shortest path from s to s)
- Set of parents comprise a $shortest paths tree$ with $O(|V|)$ size!
  (i.e., reversed shortest paths back to s from every vertex reachable from s)

Breadth-First Search (BFS)

- How to compute $\delta(s, v)$ and $P(v)$ for all $v \in V$?
- Store $\delta(s, v)$ and $P(v)$ in Set data structures mapping vertices v to distance and parent
- (If no path from s to v, do not store v in P and set $\delta(s, v)$ to ∞)
- **Idea**! Explore graph nodes in increasing order of distance
- **Goal**: Compute **level sets** $L_i = \{v | v \in V and d(s, v) = i\}$ (i.e., all vertices at distance i)
- Claim: Every vertex $v \in L_i$ must be adjacent to a vertex $u \in L_{i-1}$ (i.e., $v \in Adj(u)$)
- Claim: No vertex that is in L_j for some $j < i$, appears in L_i
- **Invariant**: $\delta(s, v)$ and $P(v)$ have been computed correctly for all v in any L_j for $j < i$
- Base case $(i = 1) : L_0 = s, \delta(s, s) = 0, P(s) = None$
- Inductive Step: To compute L_i:
  - for every vertex u in L_{i-1}:
    - for every vertex $v \in Adj(u)$ that does not appear in any L_j for $j < i$:
      - add v to L_i, set $\delta(s, v) = i$, and set $P(v) = u$
- Repeatedly compute L_i from L_j for $j < i$ for increasing i until L_i is the empty set
- Set $\delta(s, v) = \infty$ for any $v \in V$ for which $\delta(s, v)$ was not set
- Breadth-first search correctly computes all $\delta(s, v)$ and $P(v)$ by induction
- Running time analysis:
  - Store each L_i in data structure with $\Theta(|L_i|)$ time iteration and $O(1)$ time insertion (i.e., in a dynamic array or linked list)
  - Checking for a vertex v in any L_j for $j < i$ can be done by checking for v in P
  - Maintain δ and P in Set data structures supporting dictionary ops in $O(1)$ time (i.e., direct access array or hash table)
  - Algorithm adds each vertex u to ≤ 1 level and spends $O(1)$ time for each $v \in Adj(u)$
  - Work upper bounded by $O(1) \times \Sigma_{u \in V} deg(u) = O(|E|)$ by handshake lemma
  - Spend $\Theta(|V|)$ at end to assign $\delta(s, v)$ for vertices $v \in V$ not reachable from sSo
  - breadth-first search runs in linear time! $O(|V| + |E|)$

```
def bfs(adj, s):
    parent = [None for v in adj]
    parent[s] = s
```

```
    levels = [[s]]
    while 0 < len(levels[-1]):
        level = []
        for u in levels[-1]:
            for v in adj[u]:
                if parent[v] is None:
                    parent[v] = u
                    level.append(v)
        levels.append(level)
    return parents

def unweighted_shortest_path(adj, s, t):
    parents = bfs(adj, s)
    if parent[t] is None: return None
    i = t
    path = [t]
    while i != s:
        i = parent[i]
        path.append(i)
    return reversed(path)
```

10 Depth-First Search

Notes

Depth-First Search (DFS)

- Searches a graph from a vertex s, similar to BFS
- Solves Single Source Reachability, **not** Single Source Shortest Paths. Useful for solving other problems (later)!
- Return (not necessarily shortest) parent tree of parent pointers back to s.
- **Idea**! Visit outgoing adjacencies recursively, but never revisit a vertex
- i.e., follow any path until you get stuck, backtrack until finding an unexplored path to explore
- $P(s) = None$, then run $visit(s)$, where
- `visit(u)`
  - for every $v \in Adj(u)$ that does not appear in P:
    - set $P(v) = u$ and recursively call `visit(v)`
  - (DFS finishes visiting vertex u, for use later!)

```
def dfs(adj, s, parent=None, order=None):
    if parent is None:
        parent = [None for v in adj]
        parent[s] = s
        order = []
    for v in adj[s]:
        if parent[v] is None:
            parent[v] = s
            dfs(adj, v, parent, order)
    order.append(s)
    return parent, order
```

Correctness

- **Claim**: DFS visits v and correctly sets $P(v)$ for every vertex v reachable from s
- **Proof**: induct on k, for claim on only vertices within distance k from s
  - Base case $(k = 0)$: $P(s)$ is set correctly for s and s is visited
  - Inductive step: Consider vertex v with $\delta(s, v) = k' + 1$
  - Consider vertex u, the second to last vertex on some shortest path from s to v
  - By induction, since $\delta(s, u) = k'$, DFS visits u and sets $P(u)$ correctly
  - While visiting u, DFS considers $v \in Adj(u)$
  - Either v is in P, so has already been visited, or v will be visited while visiting u
  - In either case, v will be visited by DFS and will be added correctly to P $\square$

Running Time

- Algorithm `visits` each vertex u at most once and spends $O(1)$ time for each $v \in Adj(u)$
- Work upper bounded by $O(1) \times \Sigma_{u \in V} deg(u) = O(|E|)$
- Unlike BFS, not returning a distance for each vertex, so DFS runs in $O(|E|)$ time

Full-BFS and Full-DFS

- Suppose want to explore entire graph, not just vertices reachable from one vertex
- **Idea**! Repeat a graph search algorithm A on any unvisited vertex
- Repeat the following until all vertices have been visited:
  - Choose an arbitrary unvisited vertex s, use A to explore all vertices reachable from s
- We call this algorithm **Full-A**, specifically Full-BFS or Full-DFS if A is BFS or DFS
- Visits every vertex once, so both Full-BFS and Full-DFS run in $O(|V| + |E|)$ time

```
def full_dfs(adj):
    parent = [None for v in adj]
    order = []
    for v in range(len(adj)):
        if parent[v] is None:
            parent[v] = v
            dfs(adj, v, parent, order)
    return parent, order
```

DFS Edge Classification

- Consider a graph edge from vertex u to v, we call the edge a **tree edge** if the edge is part of the DFS tree (i.e. $parent[v] = u$)
- Otherwise, the edge from u to v is not a tree edge, and is either:
  - a back edge - u is a descendant of v
  - a forward edge - v is a descendant of u
  - a cross edge - neither are descendants of each other

Graph Connectivity

- An **undirected** graph is **connected** if there is a path connecting every pair of vertices
- In a **directed graph**, vertex u may be reachable from v, but v may not be reachable from u
- Connectivity is more complicated for directed graphs (we won't discuss in this class)
- `Connectivity(G)`: is undirected graph G connected?
- `Connected_Components(G)`: given undirected graph $G = (V, E)$, return partition of V into subsets $V_i \subseteq V$ (**connected components**) where each V_i is connected in G and there are no edges between vertices from different connected components

- Consider a graph algorithm A that solves Single Source Reachability
- **Claim**: A can be used to solve Connected Components
- **Proof**: Run Full-A. For each run of A, put visited vertices in a connected component $\square$

Topological Sort

- A **Directed Acyclic Graph (DAG)** is a directed graph that contains no directed cycle
- A **Topological Order** of a graph $G = (V, E)$ is an ordering f on the vertices such that: every $edge(u, v) \in E$ satisfies $f(u) < f(v)$
- **Exercise**: Prove that a directed graph admits a topological ordering if and only if it is a DAG
- How to find a topological order?
- A **Finishing Order** is the order in which a Full-DFS **finishes visiting** each vertex in G
- **Claim**: If $G = (V, E)$ is a DAG, the reverse of a finishing order is a topological order
- **Proof**: Need to prove, for every $edge(u, v) \in E$ that u is ordered before v, i.e., the visit to v finishes before visiting u. Two cases:
    - If u visited before v:
        - Before visit to u finishes, will visit v (via (u, v) or otherwise)
        - Thus the visit to v finishes before visiting u
    - If v visited before u:
        - u can't be reached from v since graph is acyclic
        - Thus the visit to v finishes before visiting u

Cycle Detection

- Full-DFS will find a topological order if a graph $G = (V, E)$ is acyclic
- If reverse finishing order for Full-DFS is not a topological order, then G must contain a cycle
- Check if G is acyclic: for each edge (u, v), check if v is before u in reverse finishing order
- Can be done in $O(|E|)$ time via a hash table or direct access array
- To return such a cycle, maintain the set of **ancestors** along the path back to s in Full-DFS
- **Claim**: If G contains a cycle, Full-DFS will traverse an edge from v to an ancestor of v
- **Proof**: Consider a cycle $(v_0, v_1, \ldots, v_k, v_0)$ in G
    - Without loss of generality, let v_0 be the first vertex visited by Full-DFS on the cycle
    - For each v_i, before visit to v_i finishes, will visit v_{i+1} and finish
    - Will consider edge (v_i, v_{i+1}), and if v_{i+1} has not been visited, it will be visited now
    - Thus, before visit to v_0 finishes, will visit v_k (for the first time, by v_0 assumption)
    - So, before visit to v_k finishes, will consider (v_k, v_0), where v_0 is an ancestor of v_k $\square$

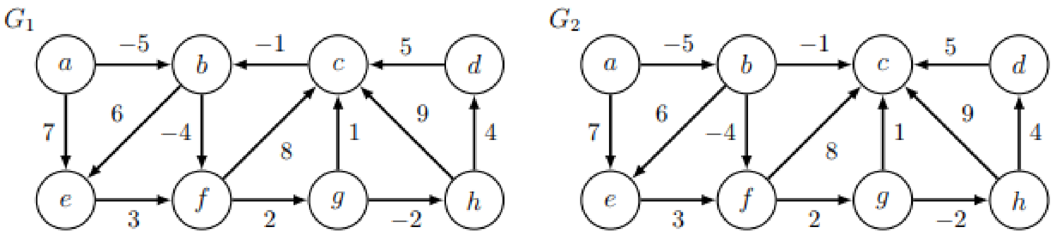
11 Weighted Shortest Paths

Notes

Weighted Graphs

- A **weighted graph** is a graph $G = (V, E)$ together with a weight function $w : E \to Z$
- i.e., assign each edge $e = (u, v) \in E$ an integer weight: $w(e) = w(u, v)$
- Many applications for edge weights in a graph:
    - distances in road network
    - latency in network connections

- strength of a relationship in a social network
- Two common ways to represent weights computationlly:
  - Inside graph representation: store edge weight with each vertex in adjacency lists
  - Store separate Set data structure mapping each edge to its weight
- We assume a representation that allows querying the weight of an edge in $O(1)$ time
- **Examples**



Weighted Paths

- The **weight** $w(\pi)$ of a path π in a weighted graph is the sum of weights of edges in the path
- The **(weighted) shortest path** from $s \in V$ to $t \in V$ is path of minimum weight from s to t
- $\delta(s,t) = inf\{w(\pi)|\ path\ \pi\ from\ s\ to\ t\}$ is the **shortest-path weight** from s to t
- (Often use "distance" for shortest -path weight in weighted graphs, not number of edges)
- As with unweighted graphs:
  - $\delta(s,t) = \inf$ if no path from s to t
  - Subpaths of shortest paths are shortest paths (or else could splice in a shortest path)
- Why infimum not minimum? Possible that no finite-length minimum-weight path exists
- When? Can occur if there is a negative-weight cycle in the graph, Ex: (b, f, g, c, b) in $G1$
- A **negative-weight cycle** is a path π starting and ending at same vertex $w(\pi) < 0$
- $\delta(s,t) = -\infty$ if there is a path from s to t through a vertex on a negative-weight cycle
- If this occurs, don't want a shortest path, but may want the negative-weight cycle

Weighted Shortest Paths Algorithms

- Already know one algorithm: Breadth-First Search! Runs in $O(|V| + |E|)$ time when, e.g.:
  - graph has positive weights, and all weights are the same
  - graph has positive weights, and sum of all weights at most $O(|V| + |E|)$
- For general weighted graphs, we don't know how to solve SSSP in $O(|V| + |E|)$ time
- But if your graph is a **Directed Acyclic Graph** you can!

Restrictions		**SSSP Algorithm**		
Graph	Weights	Name	Running Time $O(\cdot)$	
General	Unweighted	BFS	$\|V\| + \|E\|$	
DAG	Any	DAG Relaxation	$\|V\| + \|E\|$	
General	Any	Bellman-Ford	$\|V\| \cdot \|E\|$	
General	Non-negative	Dijkstra	$\|V\|log\|V\| + \|E\|$	

Shortest-Paths Tree

- For BFS, we kept track of parent pointers during search. Alternatively, compute them after!
- If know $\delta(s, v)$ for all vertices $v \in V$, can construct shortest-path tree in $O(|V| + |E|)$ time
- For weighted shortest paths from s, only need parent pointers for vertices v with finite $\delta(s, v)$
- Initialize empty P and set $P(s) = None$
- For each vertex $u \in V$ where $\delta(s, u)$ is finite:
  - For each outgoing neighbor $v \in Adj^+(u)$:
    - If $P(v)$ not assigned and $\delta(s, v) = \delta(s, u) + w(u, v)$
      - There exits a shortest path through edge (u, v), so set $P(v) = u$
- Parent pointers may traverse cycles of zero weight. Mark each vertex in such a cycle.
- For each unmarked vertex $u \in V$ (including vertices later marked):
  - For each $v \in Adj^+(u)$ where v is marked and $\delta(s, v) = \delta(s, u) + w(u, v)$
    - Unmark vertices in cycle containing v by traversing parent pointers from v
    - Set $P(v) = u$, breaking the cycle

Relaxation

- A relaxation algorithm searches for a solution to an optimization problem by starting with a solution that is not optimal, then iteratively improves the solution until it becomes an optimal solution to the original problem.

```
def try_to_relax(adj, w, d, parent, u, v):
    if d[v] > d[u] + w(u, v):
        d[v] = d[u] + w(u, v)
        parent[v] = u

def general_relax(adj, w, s):
    d = [float('inf') for _ in adj]
    parent = [None for _ in adj]
    d[s], parent[s] = 0, s
    while some_edge_relaxable(adj, w, d):
        (u, v) = get_relaxable_edge(adj, w, d)
        try_to_relax(adj, w, d, parent, u, v)
    return d, parent
```

DAG Relaxation

- **Idea**! Maintain a distance estimate $d(s, v)$ (initially ∞) for each vertex $v \in V$, that always upper bounds true distance $\delta(s, v)$, then gradually lowers until $d(s, v) = \delta(s, v)$
- When do we lower? When an edge violates the triangle inequality!
- **Triangle Inequality**: the shortest-path weight from u to v cannot be greater than the shortest path from u to v through another vertex x, i.e., $\delta(u, v) \neq \delta(u, x) + \delta(x, v)$ for all $u, v, x \in V$
- If $d(s, v) > d(s, u) + w(u, v)$ for some edge u, v, then triangle inequality is violated :(
- Fix by lowering $d(s, v)$ to $d(s, u) + w(u, v)$, i.e., **relax** (u, v) to satisfy violated constraint
- **Claim**: Relaxation is **safe**: maintains that each $d(s, v)$ is weight of a path to v (or ∞) $\forall v \in V$
- **Proof**: Assume $d(s, v')$ is weight of a path (or ∞) for $\forall v' \in V$. Relaxing some edge (u, v) sets $d(s, v)$ to $d(s, u) + w(u, v)$, which is the weight of a path from s to v through u $\square$
- Set $d(s, v) = \infty$ for all $v \in V$, then set $d(s, s) = 0$
- Process each vertex u in a topological sort order of G:
  - For each outgoing neighbor $v \in Adj^+(u)$:

- If $d(s,v) > d(s,u) + w(u,v)$
  - relax edge (u,v), i.e., set $d(s,v) = d(s,u) + w(u,v)$

```
def DAGRelaxation(adj, w, s):
    _, order = dfs(adj, s)
    d = [float('inf') for _ in adj]
    parent = [None for _ in adj]
    d[s], parent[s] = 0, s
    for u in order:
        for v in adj[u]:
            try_to_relax(adj, w, d, parent, u, v)
    return d, parent
```

Correctness

- **Claim**: At end of DAG Relaxation: $d(s,v) = \delta(s,v)$ for all $v \in V$
- **Proof**: Induct on k: $d(s,v) = \delta(s,v)$ for all v in first k vertices in topological order
  - Base case: Vertex s and every vertex before s in topological order satisfies claim at start
  - Inductive Step: Assume claim holds for first k' vertices, let v be the $(k'+1)^{th}$
  - Consider a shortest path from s to v, and let u be the vertex preceding v on path
  - u occurs before v in topological order, so $d(s,u) = \delta(s,u)$ by induction
  - When processing u, $d(s,v)$ is set to be no larger than $\delta(s,u) + w(u,v) = \delta(s,v)$
  - But $d(s,v) \geq \delta(s,v)$ since relaxation is safe, so $d(s,v) = \delta(s,v)$
- Alternatively:
  - For any vertex v, DAG relaxation sets $d(s,v) = min\{d(s,u) + w(u,v) | u \in dj^-(v)\}$
  - Shortest path to v must pass through some incoming neighbor u of v
  - So if $d(s,u) = \delta(s,u)$ for all $u \in Adj^-(v)$ by induction, then $d(s,v) = \delta(s,v)$

Running Time

- Initialization takes $O(|V|)$ time, and Topological Sort takes $O(|V| + |E|)$ time
- Additional work upper bounded by $O(1) \times \Sigma_{u \in V} deg^+(u) = O(|E|)$
- Total running time is linear, $O(|V| + |E|)$

12 Bellman-Ford

Notes

Simple Shortest Paths

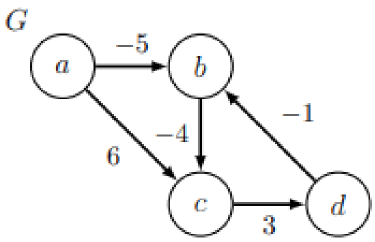
- If graph contains cycles and negative weights, might contain negative-weight cycles :(
- If graph does not contain negative-weight cycles, shortest paths are simple!
- **Claim 1**: If $\delta(s,v)$ is finite, there exists a shortest path to v that is simple
- **Proof**: By contradiction:
  - Suppose no simple shortest path: let π be a shortest path with fewest vertices
  - π not simple, so exists cycle C in π; C has non-negative weight (or else $\delta(s,v) = -\infty$)
  - Removing C form π forms path π' with fewest vertices and weight $w(\pi') \leq w(\pi)$
- Since simple paths cannot repeat vertices, finite shortest paths contain at most $|V| - 1$ edges

Negative Cycle Witness

- **k-Edge Distance** $\delta_k(s, v)$: the minimum weight of any path from s to v using $\leq k$ edges
- **Idea**! Compute $\delta_{|V|-1}(s, v)$ and $\delta_{|V|}(s, v)$ for all $v \in V$
  - If $\delta(s, v) \neq -\infty$, $\delta(s, v) = \delta_{|V|-1}(s, v)$, since a shortest path is simple (or nonexistent)
  - If $\delta_{|V|}(s, v) < \delta_{|V|-1}(s, v)$
    - there exists a shorter non-simple path to v, so $\delta_{|V|}(s, v) = -\infty$
    - call v a (negative cycle) witness
  - However, there may be vertices with $-\infty$ shortest-path weight that **are not witness**
- **Claim 2**: if $\delta(s, v) = -\infty$, then v is reachable from a witness
- **Proof**: Suffices to prove: every negative-weight cycle reachable from s contains a witness
  - Consider a negative-weight cycle C reachable from s
  - For $v \in C$, let $v' \in C$ denote v's predecessor in C, where $\Sigma_{v \in C} w(v', v) < 0$
  - Then $\delta_{|V|}(s, v) \leq \delta_{|V|-1}(s, v') + w(v', v)$ (RHS weight of some path on $\leq |V|$ vertices)
  - so $\Sigma_{v \in C} \delta_{|V|}(s, v) \leq \Sigma_{v \in C} \delta_{|V|-1}(s, v') + \Sigma_{v \in C} w(v', v) < \Sigma_{v \in C} \delta_{|V|-1}(s, v')$
  - If C contains no witness, $\delta_{|V|}(s, v) \geq \delta_{|V|-1}(s, v)$ for all $v \in C$, a contradiction $\square$

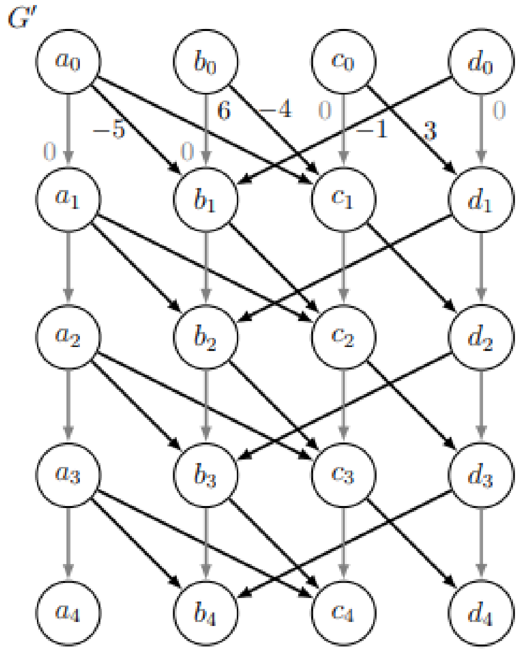
Bellman-Ford

- **Idea**! Use **graph duplication**: make multiple copies (or levels) of the graph
- $|V| + 1$ levels: vertex v_k in level k represents reaching vertex v from s using $\leq k$ edges
- If edges only increase in level, resulting graph is a DAG!
- Construct new DAG $G' = (V', E')$ from $G = (V, E)$
  - G' has $|V|(|V| + 1)$ vertices v_k for all $v \in V$ and $k \in \{0, \ldots, |V|\}$
  - G' has $|V|(|V| + |E|)$ edges:
    - $|V|$ edges (v_{k-1}, v_k) for $k \in \{1, \ldots, |V|\}$ of weight zero for each $v \in V$
    - $|V|$ edges (u_{k-1}, v_k) for $k \in \{1, \ldots, |V|\}$ of weight $w(u, v)$ for each $(u, v) \in E$
- Run DAG Relaxation on G' from s_0 to compute $\delta(s_0, v_k)$ for all $v_k \in V'$
- For each vertex: set $d(s, v) = \delta(s_0, v_{|v-1|})$
- For each witness $u \in V$ where $\delta(s_0, u_{|V|}) < \delta(s_0, u_{|V|-1})$
  - For each vertex v reachable from u in G:
    - set $d(s, v) = -\infty$



$\delta(a_0, v_k)$

$k \setminus v$	a	b	c	d
0	0	∞	∞	∞
1	0	$\mathbf{-5}$	$\mathbf{6}$	∞
2	0	-5	$\mathbf{-9}$	$\mathbf{9}$
3	0	-5	-9	$\mathbf{-6}$
4	0	$\mathbf{-7}$	-9	-6
$\delta(a, v)$	0	$-\infty$	$-\infty$	$-\infty$




```
INF = float('inf')

def bellman_ford(adj, w, s):
    # initialization
    d = [INF for _ in adj]
    parent = [None for _ in adj]
    d[s], parent[s] = 0, s
    V = len(adj)

    # construct shortest paths in rounds
    for k in range(V-1):
        for u in range(V):
            for v in adj[u]:
                try_to_relax(adj, w, d, parent, u, v)

    # check for negative weight cycles accessible from s
    for u in range(V):
        for v in adj[u]:
            if d[v] > d[u] + w(u, v):
                raise Exception("found a negative weight in cycle!")
    return d, parent
```

TBC

Running Time

- G' has size $O(|V|(|V|+|E|))$ and can be constructed in as much time
- Running DAG Relaxation on G' takes linear time in the size of G'
- Does $O(1)$ work for each vertex reachable from a witness
- Finding reachability of a witness takes $O(|E|)$ time, with at most $O(|V|)$ witnesses: $O(|V||E|)$
- (Alternatively, connect **super node** x to witnesses via 0-weight edges, linear search from x)
- Pruning G at start to only subgraph reachable from s yields $O(|V||E|)$ time algorithm

13 Dijkstra's Algorithm

Notes

Non-negative Edge Weights

- **Idea**! Generalize BFS approach to weighted graphs:
  - Grow a sphere centered at source s
  - Repeatedly explore closer vertices before further ones
  - But how to explore closer vertices if you don't know distances beforehand? :(
- **Observation 1**: If weights non-negative, monotonic distance increasing along shortest paths
  - i.e., if vertex u appears on a shortest path from s to v, then $\delta(s,u) \le \delta(s,v)$
  - Let $V_x \subset V$ be the subset of vertices reachable within distance $\le x$ from s
  - If $v \in V_x$ then any shortest path from s to v only contains vertices from V_x
  - Perhaps grow V_x one vertex at a time! (but growing for every x is slow if weights large)
- **Observation 2**: Can solve SSSP fast if given order of vertices in increasing distance from s
  - Remove edges that go against this order (since cannot participate in shortest paths)
  - May still have cycles if zero-weight edges: repeatedly collapse into single vertices
  - Compute $\delta(s,v)$ for each $v \in V$ using DAG relaxation in $O(|V| + |E|)$ time

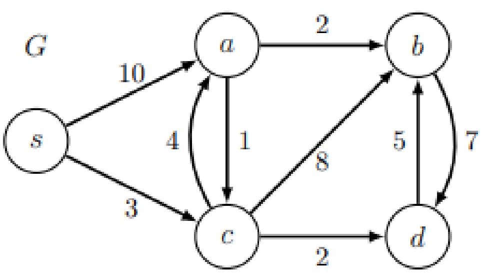
Dijkstra's Algorithm

- **Idea**! Relax edges from each vertex in increasing order of distance from source s
- **Idea**! Efficiently find next vertex in the order using a data structure
- **Changeable Priority Queue** Q on items with keys and unique IDs, supporting operations:

Operation	Specification
`Q.build(X)`	initialize Q with items in iterator `X`
`Q.delete_min()`	remove an item with minimum key
`Q.decrease_key(id, k)`	find stored item with ID `id` and change key to `k`

- Implement by **cross-linking** a Priority Queue Q' and a Dictionary D mapping IDs into Q'
- Assume vertex IDs are integers from 0 to $|V| - 1$ so can use a direct access array for D
- For brevity, say item x is the tuple $(x.id, x.key)$
- Set $d(s,v) = \infty$ for all $v \in V$, then set $d(s,s) = 0$
- Build changeable priority queue Q with an item $(v, d(s,v))$ for each vertex $v \in V$
  - For vertex v in outgoing adjacencies $Adj^+(u)$:
    - If $d(s,v) > d(s,u) + w(u,v)$:
      - Relax edge (u,v), i.e., set $d(s,v) = d(s,u) + w(u,v)$
      - Decrease the key of v in Q to new estimate $d(s,v)$

Delete v from Q	s	a	b	c	d
s	$\mathbf{0}$	∞	∞	∞	∞
c		10	∞	$\mathbf{3}$	∞
d		7	11		$\mathbf{5}$
a		$\mathbf{7}$	10		
b			$\mathbf{9}$		
$\delta(s,v)$	0	7	9	3	5



```python
def dijkstra(adj, w, s):
    d = [INF for _ in adj]
    parent = [None for _ in adj]
    d[s], parent[s] = 0, s
    Q = PriorityQueue()
    V = len(adj)
    for v in range(V):
        Q.insert(v, d[v])   # label and key
    for _ in range(V):
        u = Q.extract_min() # get label for item with min key
        for v in adj[u]:
            try_to_relax(adj, w, d, parent, u, v)
            Q.decrease_key(v, d[v]) # alter key for given label
    return d, parent


class PriorityQueue:
    def __init__(self):
        self.A = {}

    def insert(self, label, key):
        self.A[label] = key

    def extract_min(self):
        min_label = None
        for label in self.A:
            if (min_label is None) or (self.A[label] < self.A[min_label]):
                min_label = label
        del self.A[min_label]
        return min_label

    def decrease_key(self, label, key):
        if (label in self.A) and (key < self.A[label]):
            self.A[label] = key


class Item:
    def __init__(self, label, key):
        self.label, self.key = label, key

    def __lt__(self, other):
        return self.key < other.key

class PriorityQueue:
    def __init__(self):
        self.A = []
        self.label_2_idx = dict()

    def insert(self, label, key):
        item = Item(label, key)
        self.A.append(item)
        idx= len(self.A) - 1
        self.label_2_idx[label] = idx
        heapq._siftdown(self.A, 0, idx)

    def extract_min(self):
        label = self.A[0].label
        self.A[0], self.A[-1] = self.A[-1], self.A[0]
        self.label_2_idx[self.A[0].label] = 0
        self.label_2_idx.pop(self.A[-1].label)
        self.A.pop()
        if self.A: heapq._siftup(self.A, 0)
```

```python
        return label

    def decrease_key(self, label, key):
        if label in self.label_2_idx:
            idx = self.label_2_idx[label]
            if self.A[idx].key < key:
                self.A[idx].key = key
                heapq._siftdown(self.A, 0, idx)
```

Correctness

- **Claim**: At end of Dijkstra's algorithm $d(s, v) = \delta(s, v)$ for all $v \in V$
- **Proof**:
  - If relaxation sets $d(s, v)$ to $\delta(s, v)$, then $d(s, v) = \delta(s, v)$ at the end of the algorithm
    - Relaxation can only decrease estimates $d(s, v)$
    - Relaxation is safe, i.e., maintains that each $d(s, v)$ is weight of a path to v (or ∞)
  - Suffices to show $d(s, v) = \delta(s, v)$ when vertex v is removed from Q
    - Proof by induction on first k vertices removed from Q
    - Base Case (k = 1): s is first vertex removed from Q, and $d(s, s) = 0 = \delta(s, s)$
    - Inductive Step: Assume true for $k < k'$, consider k' th vertex v_0 removed from Q
    - Consider some shortest path π from s to v', with $w(\pi) = \delta(s, v')$
    - Let (x, y) be the first edge in π where y is not among first $k' - 1$ (perhaps $y = v'$)
    - When x was removed from Q, $d(s, x) = \delta(s, x)$ by induction, so:

$$\begin{aligned} d(s, y) &\leq \delta(s, x) + w(x, y) \\ &= \delta(s, y) \\ &\leq \delta(s, v') \\ &\leq d(s, v') \\ &\leq d(s, y) \end{aligned}$$

    - So $d(s, v') = \delta(s, v')$ as desired

Running Time

- Count operations on changeable priority queue Q, assuming it contains n items:

Operation	Time	Occurrences in Dijkstra
`Q.build(X)`	B_n	1
`Q.delete_min()`	M_n	|V|
`Q.decrease_key(id, k)`	D_n	|E|

- Total running time is $O(B_{|V|} + |V| \cdot M_{|V|} + |E| \cdot D_{|V|})$
- Assume pruned graph to search only vertices reachable from the source, so $|V| = O(|E|)$

TBC

15 Recursive Algorithms

Notes

Design your own recursive algorith

- Constant-sized program to solve arbitrary input
- Need looping or recursion, analyze by induction
- Recursive function call: vertex in a graph, directed edge from $A \to B$ if B calls A
- Dependency graph of recursive calls must be acyclic (if can terminate)
- Classify based on shape of graph

Class	Graph
Brute Force	Star
Decrease & Conquer	Chain
Divide & Conquer	Tree
Dynamic Programming	DAG
Greedy / Incremental	Subgraph

- Hard part is thinking inductively to construct recurrence on subproblems
- How to solve a problem recursively (**SRT BOT**)
  - **Subproblem** definition
  - **Relate** subproblem solutions recursively
  - **Topological** order on subproblems ($\Rightarrow$ subproblem DAG)
  - **Base** cases of relation
  - **Original** problem solution via subproblems(s)
  - **Time** analysis

Merge Sort in SRT BOT Framework

- Merge sorting an array A of n elements can be expressed in SRT BOT as follows:
- **S**ubproblems: $S(i, j)$ = sorted array on elements of $A[i : j]$ for $i \leq i \leq j \leq n$
- **R**elation: $S(i, j) = merge(S(i, m), S(m, j))$ where $m = \lfloor (i + j)/2 \rfloor$
- **T**opological order: Increasing $j - i$
- **B**ase cases: $S(i, i + 1) = [A[i]]$
- **O**riginal: $S(0, n)$
- **T**ime: $T(n) = 2T(n/2) + O(n) = O(nlogn)$

Fibonacci Numbers

- Compute the nth Fibonacci number F_n
- **S**ubproblems: $F(i)$ = the ith Fibonacci number F_i for $i \in \{0, 1, \ldots, n\}$
- **R**elation: $F(i) = F(i - 1) + F(i - 2)$ (definition of Fibonacci numbers)
- **T**opological order: Increasing i
- **B**ase cases: $F(0) = 0, F(1) = 1$
- **O**riginal problem: $F(n)$

```
def fib(n):
    if n < 2: return n
    return fib(n-1) + fib(n-2)
```

- Divide and conquer implies a tree of **recursive calls**
- **T**ime: $T(n) = T(n - 1) + T(n - 2) + O(1) > 2T(n - 2), T(n) = \Omega(2^{n/2})$ exponential... :(
- Subproblem $F(k)$ computed more than once! ($F(n - k)$ times)
- Can we avoid this waste?

Re-using Subproblem Solutions

- Either:
  - **Top down**: record subproblem solutions in a memo and re-use
  - **Bottom up**: solve subproblems in topological sort order (usually via loops)
- For Fibonacci, $n+1$ subproblems (vertices) and $< 2n$ dependencies (edges)
- Time to compute is then $O(n)$ additions

```
def fib(n):
    memo = dict()
    def F(i):
        if i < 2: return i
        if i not in memo:
            memo[i] = F(i-1) + F(i-2)
        return memo[i]
   return F(n)

def fib(n):
    F = dict()
    F[0], F[1] = 0, 1
    for i in range(2, n+1):
        F[i] = F[i-1] + F[i-2]
    return F[n]
```

- A subtlety is that Fibonacci numbers grow to $\Theta(n)$ bits long, potentially $\gg$ word size w
- Each addition costs $O(\lceil n/w \rceil)$ time
- So total cost is $O(n \lceil n/w \rceil) = O(n + n^2/w)$ time

Dynamic Programming

- Weird name coined by Richard Bellman
  - Wanted government funding, needed cool name to disguise doing mathematics!
  - Updating (dynamic) a plan or schedule (program)
- Existence of recursive solution implies decomposable subproblems
- Recursive algorithm implies a graph of computation
- Dynamic programming if subproblem dependencies **overlap** (DAG, in-degree > 1)
- "Recurse but re-use" (Top down: record and lookup subproblem solutions)
- "Careful brute force" (Bottom up: do each subproblem in order)
- Often useful for **counting/optimization** problems: almost trivially correct recurrences

How to Solve a Problem Recursively (SRT BOT)

- **Subproblem** definition subproblem $x \in X$
  - Describe the meaning of a subproblem **in words**, in terms of parameters
  - Often subsets of input: prefixes, suffixes, contiguous substrings of a sequence
  - Often record partial state: add subproblems by incrementing some auxiliary variable
- **Relate** subproblem solutions recursively $x(i) = f(x(j), \dots)$ for one or more $j < i$
- **Topological order**: to argue relation is acyclic and subproblems form a DAG
- **Base** cases
  - State solutions for all (reachable) independent subproblems where relation breaks down
- **Original problem**
  - Show how to compute solution to original problem from solutions to subproblem(s)
  - Possibly use parent pointers to recover actual solution, not just objective function

- **Time** analysis
  - $\Sigma_{x \in X} work(x)$, or if $work(x) = O(W)$ for all $x \in X$, then $|X| \cdot O(W)$
  - $work(x)$ measures non-recursive work in relation; treat recursions as taking $O(1)$ time

DAG Shortest Paths

- DAG SSSP problem: given a DAG G and vertex s, compute $\delta(s, v)$ for all $v \in V$
- **S**ubproblems: $\delta(s, v)$ for all $v \in V$
- **R**elation: $\delta(s, v) = min\{\delta(s, u) + w(u, v) | u \in Adj^-(v)\} \cup \{\infty\}$
- **T**opological order: Topological order of G
- **B**ase case: $\delta(s, s) = 0$
- **O**riginal: All subproblem
- **T**ime: $\Sigma_{v \in V} O(1 + |Adj^-(v)|) = O(|V| + |E|)$
- DAG Relaxation computes the same min values as this dynamic program, just
  - step-by-step (if new value < min, update min via edge relaxation), and
  - from the perspective of u and $Adj^+(u)$ instead of v and $Adj^-(v)$

How to Relate Subproblem Solutions

- The general approach we're following to define a relation on subproblem solutions:
  - Identify a question about a subproblem solution that, if you knew the answer to, would reduce to "smaller" subproblem(s)
  - Then locally brute-force the question by trying all possible answers, and taking the best
  - Alternatively, we can think of correctly guessing the answer to the question, and directly recursing; but then we actually check all possible guesses, and return the "best"
- The key for efficiency is for the question to have a small (polynomial) number of possible answers, so brute forcing is not too expensive
- Often (but not always) the non-recursive work to compute the relation is equal to the number of answers we're trying

16 Dynamic Programming Subproblems

Notes

Longest Common Subsequence (LCS)

- Given two strings A and B, find a longest (not necessarily contiguous) subsequence of A that is also a subsequence of B.
- Example: $A = hieroglyphology$ $B = michaelangelo$
- Solution: $hello$ or $heglo$ or $iello$ or $ieglo$, all length 5
- Maximization problem on length of subsequence

1. **S**ubproblems:
   - $x(i, j) =$ length of the longest common subsequence of suffixes $A[i:]$ and $B[j:]$
   - For $0 \le i \le |A|$ and $0 \le j \le |B|$
2. **R**elate:
   - Either first characters match or they don't
   - If first characters match, some longest common subsequence will use them
   - (if no LCS uses first matched pair, using it will only improve solution)
   - (if an LCS uses first in $A[i]$ but not first in $B[j]$, matching $B[j]$ is also optimal)
   - If they do not match, they cannot both be in a longest common subsequence
   - Guess whether $A[i]$ or $B[j]$ is not in LCS

- $x(i, j) = \begin{cases} x(i+1, j+1) + 1 & if A[i] = B[j] \\ max\{x(i+1, j), x(i, j+1)\} & otherwise \end{cases}$

3. **T**opological order:
   - Subproblem $x(i, j)$ depend only on strictly larger i or j or both
   - Simplest order to state: Decreasing $i + j$
   - Nice order for bottom-up code: Decreasing i, then decreasing j
4. **B**ase
   - $x(i, |B|) = x(|A|, j) = 0$ (one string is empty)
5. **O**riginal problem
   - Length of longest common subsequence of A and B is $x(0, 0)$
   - Store parent pointers to reconstruct subsequencce
   - If the parent pointer increases both indices, add that character to LCS
6. **T**ime:
   - # subproblems: $(|A| + 1) \cdot (|B| + 1)$
   - work per subproblem: $O(1)$
   - $O(|A| \cdot |B|)$ running time

```
def lcs(A, B):
    a, b = len(A), len(B)
    x = [[0] * (b + 1) for _ in range(a + 1)]
    for i in reversed(range(a)):
        for j in reversed(range(b)):
            if A[i] == B[j]:
                x[i][j] = x[i + 1][j + 1] + 1
            else:
                x[i][j] = max(x[i + 1][j], x[i][j + 1])
    return x[0][0]

def lcs(A, B):
    a, b = len(A), len(B)
    x = [[0] * (b + 1) for _ in range(a + 1)]
    for i in range(1, a + 1):
        for j in range(1, b + 1):
            if A[i] == B[j]:
                x[i][j] = x[i - 1][j - 1] + 1
            else:
                x[i][j] = max(x[i - 1][j], x[i][j - 1])
    return x[0][0]
```

Longest Increasing Subsequence (LIS)

- Given a string A, find a longest (not necessarily contiguous) subsequence of A that strictly increases (lexicographically).
- Example: $A = carbohydrate$
- Solution: $abort$ of length 5
- Maximization problem on length of subsequence
- Attempted solution:
   - Natural subproblems are prefixes or suffixes of A, say suffix $A[i:]$
   - Natural question about LIS of $A[i:]$: is $A[i]$ in the LIS? (2 possible answers)
   - But then how do we recurse on $A[i+1:]$ and guarantee increasing subsequence?
   - **Fix**: add **constraint** to subproblems to give enough structure to achieve increasing property

1. Subproblems
   - $x(i)$ = length of longest increasing subsequence of suffix $A[i:]$ that includes $A[i]$
   - For $0 \leq i \leq |A|$

2. Relate
    - We're told that $A[i]$ is in LIS (first element)
    - Next question: what is the second element of LIS?
        - Could be any $A[j]$ where $j > i$ and $A[j] > A[i]$ (so increasing)
        - Or $A[i]$ might be the last element of LIS
    - $x(i) = max\{1 + x(j) | i < j < |A|, A[j] > A[i]\} \cup \{1\}$
3. Topological order:
    - Decreasing i
4. Base
    - No base case necessary, because we consider the possibility that $A[i]$ is last
5. Original problem
    - What is the first element of LIS? **Guess**!
    - Length of LIS of A is $max\{x(i) | 0 \le i < |A|\}$
    - Store parent pointers to reconstruct subsequence
6. Time
    - # subproblems: $|A|$
    - work per subproblem $O(|A|)$
    - $O(|A|^2)$ running time
    - speed up to $O(|A| log |A|)$ by doing only $O(log |A|)$ work per subproblem, via AVL tree augmentation

```
def lis(A):
    a = len(A)
    x = [1] * a
    for i in reversed(range(a)):
        for j in range(i, a):
            if A[j] > A[i]:
                x[i] = max(x[i], 1 + x[j])
    return max(x)
```

Alternating Coin Game

- Given sequence of n coins of value $v_0, v_1, \ldots, v_{n1}$
- Two players (“me” and “you”) take turns
- In a turn, take first or last coin among remaining coins
- My goal is to maximize total value of my taken coins, where I go first
- First solution exploits that this is a zero-sum game: I take all coins you don’t

1. Subproblems
    - Choose subproblems that correspond to the state of the game
    - For every contiguous subsequence of coins from i to j, $0 \le i \le j < n$
    - $x(i, j) =$ maximum total value I can take starting from coins of values $v_i, \ldots, v_j$
2. Relate
    - I must choose either coin i or coin j (**Guess**!)
    - Then it's your turn, so you'll get values $x(i + 1, j)$ or $x(i, j - 1)$ respectively
    - To figure out how much value I get, subtract this from total coin values
    - $x(i, j) = max\{v_i + \Sigma_{k=i+1}^{j} v_k \quad x(i + 1, j), v_j + \Sigma_{k=i}^{j} v_k \quad x(i, j - 1)\}$???
3. Topological order
    - Increasing $j - i$
4. Base
    - $x(i, i) = v_i$
5. Original problem

    - $x(0, n-1)$
    - store parent pointers to reconstruct strategy
6. Time
    - # subproblems: $\Theta(n^2)$
    - work per subproblem: $\Theta(n)$ to compute sums
    - $\Theta(n^3)$ running time
    - Speed up to $\Theta(n^2)$ time by pre-computing all sums $\Sigma_{k=i}^{j} v_k$ in $\Theta(n^2)$ time via dynamic programming

- Second solution uses **subproblem expansion**: add subproblems for when you move next

1. Subproblems
    - Choose subproblems that correspond to the full state of the game
    - Contiguous subsequence of coins from i to j, and which player p goes next
    - $x(i, j, p) =$ maximum total value I can take when player $p \in \{me, you\}$ starts from coins of values $v_i, \ldots, v_j$
2. Relate
    - Player p must choose either coin i or coin j (**Guess**!)
    - If $p =$ me, then I get the value; otherwise, I get nothing
    - Then it's the other player's turn
    - $x(i, j, me) = max\{v_i + x(i+1, j, you), v_j + x(i, j-1, you)\}$
    - $x(i, j, you) = min\{x(i+1, j, me), x(i, j-1, me)\}$
3. Topological order
    - Increasing $j - i$
4. Base
    - $x(i, i, me) = v_i$
    - $x(i, i, you) = 0$
5. Original problem
    - $x(0, n-1, me)$
    - Store parent pointers to reconstruct strategy
6. Time
    - # subproblems: $\Theta(n^2)$
    - work per subproblem: $\Theta(1)$
    - $\Theta(n^2)$ running time

Yet another alternative solution.

```
def coin_game(coins):
    n = len(coins)
    dp = [[0] * n for _ in range(n)]
    for i in reversed(range(n)):
        for j in range(i, n):
            if i == j:
                d[i][j] = coins[i]
            else:
                dp[i][j] = max(coins[i] - dp[i+1][j], coins[j] - dp[i][j-1])
    return dp[0][n-1] >= 0

def coin_game(coins):
    n = len(coins)
    dp = [[0] * n for _ in range(n)]
    parents = dict()

    for i in reversed(range(n)):
        for j in range(i, n):
            if i == j:
```

```
                d[i][j] = coins[i]
                parents[(l, r)] = ((l, r), coins[l])
            else:
                a = coins[i] - dp[i+1][j]
                b = coins[j] - dp[i][j-1]
                if a > b:
                    dp[i][j] = a
                    parents[(l, r)] = ((l+1, r), nums[l])
                else:
                    dp[i][j] = b
                    parents[(l, r)] = ((l, r-1), nums[r])

    if dp[0][n-1] >= 0
        state = (0, n-1)
        turn = 1
        while parents[state][0] != state:
            print(f"player {turn % 2} took {parents[state][1]}")
            state = parents[state][0]
            turn += 1

    return dp[0][n-1] >= 0
```

Subproblem Constraints and Expansion

- We've now seen two examples of constraining or expanding subproblems
- If you find yourself lacking information to check the desired conditions of the problem, or lack the natural subproblem to recurse on, try subproblem constraint/expansion!
- More subproblems and constraints give the relation more to work with, so can make DP more feasible
- Usually a trade-off between number of subproblems and branching/complexity of relation

17 Dynamic Programming III

Notes

Single-Source Shortest Paths Revisited

1. Subproblems
    - Expand subproblems to add information to make acyclic!
    - $\delta_k(s, v) =$ weight of shortest path from s to v using at most k edges
    - For $v \in V$ and $0 \leq k \leq |V|$
2. Relate:
    - Guess last edge (u, v) on shortest path from s to v
    - $\delta_k(s, v) = min\{\delta_{k-1}(s, u) + w(u, v) | (u, v) \in E\} \cup \{\delta_{k-1}(s, v)\}$
3. Topological order:
    - Increasing k: subproblems depend on subproblems only with strictly smaller k.
4. base
    - $\delta_0(s, s) = 0$ and $\delta_0(s, v) = \infty$ for $v \neq s$ (no edges)
5. Original problem
    - If has finte shortest path, then $\delta(s, v) = \delta_{|V|-1}(s, v)$
    - Otherwise some $\delta_{|V|}(s, v) < \delta_{|V|-1}(s, v)$, so path contains a negative-weight cycle
    - Can keep track of parent pointers to subproblem that minimized recurrence
6. Time
    - # subproblems: $|V| \times (|V| + 1)$
    - Work for subproblem $\delta_k(s, v) : O(deg_{in}(v))$

$$\Sigma_{k=0}^{|V|}\Sigma_{v\in V}O(deg_{in}(v)) = \Sigma_{k=0}^{|V|}O(|E|) = O(|V|\cdot|E|)$$

- This is just Bellman-Ford! (computed in a slightly different order)

All-Pairs Shortest Paths: Floyd-Warshall

- Could define subproblem $\delta_k(u, v) =$ minimum weight of path from u to v using at most k edges, as in Bellman-Ford
- Resulting running time is $|V|$ times Bellman-Ford, i.e., $O(|V|^2 \cdot |E|) = O(|V|^4)$
- Know a better algorithm from L14: Johnson achieves $O(|V|^2 log|V + |V| \cdot |E|) = O(|V|^3)$
- Can achieve $\Theta(|V|^3)$ running time (matching Johnson for dense graphs) with a simple dynamic program, called **Floyd-Warshall**.
- Number vertices so that $V = \{1, \ldots, |V|\}$

1. Subproblems:
    - $d(u, v, k) =$ minimum weight of a path from u to v that only uses vertices from $\{1, 2, \ldots, k\} \cup \{u, v\}$
    - For $u, v \in V$ and $1 \leq k \leq |V|$
2. Relate
    - $x(u, v, k) = min\{x(u, k, k-1) + x(k, v, k-1), x(u, v, k-1)\}$
    - Only constant branching! No longer guessing previous vertex/edge
3. Topological order
    - Increasing k: relation depends only on smaller k
4. Base
    - $x(u, u, 0) = 0$
    - $x(u, v, 0) = w(u, v)$ if $(u, v) \in E$
    - $x(u, v, 0) = \infty$ if none of the above
5. Original problem
    - $x(u, v, |V|)$ for all $u, v \in V$
6. Time
    - $O(|V|^3)$ subproblems
    - Each $O(1)$ work
    - $O(|V|^3)$ in total
    - Constant number of dependencies per subproblem brings the factor of $O(|E|)$ in the running time down to $O(|V|)$